Optimal distributed control with application to asymmetric vehicular platoons

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***Abstract*— This paper considers a distributed system of identical agents with arbitrary models. A method for distributed state-feedback design is provided. The proposed solution consists of two steps: first a single-agent controller is derived and then, based on the network topology, the gain of this controller is adjusted. LQ optimality of this controller is proved provided that the Laplacian has only real eigenvalues and is non-defective. The result is subsequently used to design a controller for asymmetric vehicular platoon. We show that the same controller with fixed gain is the optimal controller for any number of vehicles in the platoon. However, the performance of the optimal controller is still subject to inherent limitations given by the network topology. In some cases, even exponential scaling in the number of vehicles must occur for any controller.**

I. Introduction

Vehicular platoons are now an intensive field of research. The reason is the possibility to increase the safety and throughput at the same time.

There are many approaches for platoon control. The most appealing is the fixed-distance scenario, in which a vehicle is required to keep a fixed distance to its nearest neighbors. The reason for its popularity is that in this case the potential to increase the throughput is the highest. The strategy easiest to implement is the predecessor following. It was shown to be string unstable in [1]. String instability, roughly speaking, means that a disturbance acting at one vehicle is amplified as it propagates in the formation. Disturbance propagation and scaling are issues also for other strategies.

Except for the nearest neighbor in front of a vehicle also the nearest neighbor behind can be used for control. This scheme is called bidirectional. Symmetric bidirectional control was investigated in [2]. This type of control suffers from long transients. Transient time can be improved using asymmetry, which means that the vehicle pays more attention to the car ahead than to the car behind. Asymmetry achieves a uniform bound on eingevalues in some cases [3]–[5], which improves the transient time a lot. Nevertheless, the price to pay might be an exponential scaling of the $\mathcal{H}_\infty$ norm of the transfer functions in platoon when increasing the number of vehicles [4]–[6]. To improve transients, different asymmetries in position and velocity were introduced [7]. Inverse optimality of a controller for a platoon is considered in [8] for a mass-spring-damper models.

The papers mentioned above mainly considered qualitative behavior of vehicular platoons. Not many papers provide a general method how to design a controller for the platoon. The first requirement on the platoon control is stability. Distributed control literature provides a lot of very useful approaches for achieving stability. It was shown in [9] that stability of a formation depends on the agent model and eigenvalues of the graph Laplacian. This fact was later used in [10] to derive a state-feedback controller, which works for any graph topology. Only the gain of the controller has to be adjusted. A controller design presented in [11] uses an LQR-like approach to achieve stability. Adaptive approaches can also be used [12], [13].

Regarding performance, for undirected graphs there are results presented in [14] for $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. Inverse optimality of localized static-state feedback is proved in [15] for directed graphs. The main idea of the paper is to design a static feedback based on single-agent model. When a sufficiently large coupling gain is used and certain criteria are met, the control law will not only be stabilizing, but also LQ optimal.

In this paper we are going to use the results of [15] to derive very simple conditions for optimal LQ control. They are very easy extensions of the results for synchronizing region. Ussuming pinning control, the result holds for graph Laplacian which is non-defective and has only real eigenvalues. Such control law is then applied to controller design for asymmetric vehicular platoons. We show that a controller designed using our approach is the optimal controller and that it does not change with growing number of vehicles. Next we discuss properties of the controller obtained this way. We prove uniform boundedness of eigenvalues from zero, which implies a good convergence time. On the other hand, not even an optimal controller can break the inherent limitations of the network structure. We prove that for two integrators in the open loop the $\mathcal{H}_\infty$ norm of a transfer function in platoon scales exponentially with the graph distance between nodes. As we show by simulations, the scaling of a system with one integrator depends on the controller parameters.

II. System model

Consider N identical agents modelled as LTI systems

$$\begin{aligned}\dot{x}_i &= Ax_i + Bu_i\\ y_i &= Cx_i\end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}$, $x_i \in \mathbb{R}^n$ is the state vector of the ith agent, $u_i \in \mathbb{R}^m$ is the control input of the ith agent and $y_i \in \mathbb{R}^p$ is its output. We assume that the pair (A, B) is stabilizable. We denote υ the number of the eigenvalues of A at the origin—this number is also known as the Type number of the system or the number of integrators

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in the open loop. The overall multi-agent system is

$$\begin{aligned} \dot{x} &= (I \otimes A)x + (I \otimes B)u \\ y &= (I \otimes C)x, \end{aligned} \tag{2}$$

where $x = [x_1^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}}]^{\mathrm{T}}$, $u = [u_1^{\mathrm{T}}, \ldots, u_N^{\mathrm{T}}]^{\mathrm{T}}$ and $y = [y_1^{\mathrm{T}}, \ldots, y_N^{\mathrm{T}}]^{\mathrm{T}}$.

The agents are interconnected using a relative static state feedback of the form

$$u_i = \sum_{j=\mathcal{N}_i} cKl_{ij}(x_j - x_i), \tag{3}$$

where $\mathcal{N}$ is a set of neighbors of the agent i, $l_{ij} > 0$ is a weight of the coupling between agents i and j, $K \in \mathbb{R}^{m\times n}$ is a static-state feedback matrix and c is a coupling gain. We will show how to optimally design K and c.

We also assume that there is a leader present in the system. This leader serves as a reference available only to some of the agents. The control law (3) is then modified to

$$u_i = \sum_{j=\mathcal{N}_i} cK\Big[l_{ij}(x_j - x_i) + p_i(\rho - x_i)\Big], \tag{4}$$

where $p_i \geq 0$ is a weight of the coupling between the leader and the ith agent and ρ is a state of the leader to which the agents should synchronize. This control scheme is called a pinning control [11].

In vector form, the control input can be written as

$$u = -[(L + P) \otimes cK]x \tag{5}$$

where $P = \mathrm{diag}[p_1, p_2, \ldots, p_N]$ and L is the graph Laplacian, capturing the interconnection. The Laplacian is defined as $L = D - E$, where $E = [l_{ij}]$ is the adjacency matrix of the graph and $D = \mathrm{diag}[d_1, d_2, \ldots, d_N]$ with d_i being the in-degree of the agent i. If the graph corresponding to L contains a directed spanning tree and the leader pins to the root of this tree, then the pinned Laplacian $L_{\mathrm{p}} = L + P$ is a nonsingular matrix [16]. We denote the ith eigenvalue of L_{p} as λ_i and $\lambda_{\min} > 0$ is a lower bound on eigenvalues, i.e., $\lambda_{\min} \leq |\lambda_i|\, \forall i$.

Plugging (5) to (2), we get the model of the overall multi-agent system

$$\begin{aligned} \dot{x} &= (I \otimes A)x - (L_{\mathrm{p}} \otimes cBK)x \\ y &= (I \otimes C)x. \end{aligned} \tag{6}$$

We denote the ith eigenvalue of the closed-loop feedback matrix $(I \otimes A) - (L_{\mathrm{p}} \otimes cBK)$ as ν_i.

III. Controller design

In this section we show how to obtain the feedback matrix K and the coupling gain c for a special case. The feedback matrix K will be designed using a standard LQR procedure [11], [15] for individual agent. Consider the criterion

$$J_i(x_i, u_i) = \int_0^\infty x_i^{\mathrm{T}} Q_{\mathrm{a}} x_i + u_i^{\mathrm{T}} R_{\mathrm{a}} u_i \,\mathrm{d}t, \tag{7}$$

where $Q_{\mathrm{a}} \geq 0, R_{\mathrm{a}} > 0$ (we use $A > 0$ ($A \geq 0$) with a meaning that A is symmetric and positive definite (semi-definite)). Then the control law of individual agent is $u_i = -cKx_i$ with

$$K = R_{\mathrm{a}}^{-1} B^{\mathrm{T}} P_{\mathrm{a}}, \tag{8}$$

where $P_{\mathrm{a}} > 0$ is the positive definite solution of the continuous-time algebraic Riccati equation

$$A^{\mathrm{T}} P_{\mathrm{a}} + P_{\mathrm{a}} A - P_{\mathrm{a}} B^{\mathrm{T}} R_{\mathrm{a}}^{-1} B P_{\mathrm{a}} + Q_{\mathrm{a}} = 0. \tag{9}$$

As was discussed in [11], this control law achieves an unbounded synchronization region, that is, the matrix pencil $A - \gamma BK$, $\gamma \in \mathbb{C}$ has eigenvalues in the left half-plane if $\Re\{\gamma\} > 0.5$, no matter what the imaginary part of γ is.

A sufficient condition for stability of the distributed feedback control law (6) was presented in [11] as $c \geq \frac{1}{2\Re\{\lambda_{\min}\}}$ with K designed by (8). However, this guarantees only stability, it does not take into account any performance measure of the interconnected system.

A condition, under which the local control law is optimal with respect to some performance criterion, was presented in [15]. Here we extend the result to a very simple sufficient condition for the control law design.

Theorem 1. *Assume that L_{p} is non-singular, non-defective and has only real eigenvalues, i.e., $L_{\mathrm{p}} V = V\Lambda$, Λ is real and diagonal and Λ^{-1} exists. Let $\lambda_{\min}$ be selected such that $0 < \lambda_{\min} \leq \lambda_i\, \forall i$. Then the local static state feedback control law (5) with $c \geq \frac{1}{\lambda_{\min}}$ is the optimal control law with respect to the performance criterion*

$$J(x, u) = \int_0^\infty x^{\mathrm{T}} \overline{Q} x + u^{\mathrm{T}} \overline{R} u \,\mathrm{d}t \tag{10}$$

with

$$\begin{aligned} \overline{Q} &= c^2 (L_{\mathrm{p}} \otimes K)^{\mathrm{T}} (R_{\mathrm{b}} \otimes R_{\mathrm{a}})(L_{\mathrm{p}} \otimes K) \\ &\quad - cR_{\mathrm{b}} L_{\mathrm{p}} \otimes (A^{\mathrm{T}} P_{\mathrm{a}} + P_{\mathrm{a}} A^{\mathrm{T}}) \end{aligned} \tag{11}$$

$$\overline{R} = R_{\mathrm{b}} \otimes R_{\mathrm{a}} \tag{12}$$

$$R_{\mathrm{b}} = (V^{-1})^{\mathrm{T}} V^{-1}, \tag{13}$$

for some $R_{\mathrm{a}} > 0, Q_{\mathrm{a}} \geq 0$ and $K = R_{\mathrm{a}}^{-1} B^{\mathrm{T}} P_{\mathrm{a}}$ with $P_{\mathrm{a}} > 0$ satisfying (9).

Proof: We need to satisfy the conditions of [15, Thm. 2]. The conditions are that $\overline{Q} \geq 0$ and that there exits matrix $R_{\mathrm{b}} > 0$ such that $P_{\mathrm{b}} = cR_{\mathrm{b}} L_{\mathrm{p}}$ with $P_{\mathrm{b}} > 0$. The control law assumed there is (5) and the feedback matrix K is designed using (8).

First we test if $P_{\mathrm{b}} = cR_{\mathrm{b}} L_{\mathrm{p}}$ is positive definite. Thus, let $R_{\mathrm{b}} = (V^{-1})^{\mathrm{T}} V^{-1}$ as in (13). Then

$$P_{\mathrm{b}} = cR_{\mathrm{b}} L_{\mathrm{p}} = c(V^{-1})^{\mathrm{T}} V^{-1} L_{\mathrm{p}} = c(V^{-1})^{\mathrm{T}} \Lambda V^{-1} \tag{14}$$

Based on the assumptions, $\Lambda > 0$ (the eigenvalues of L_{p} are positive) and it follows that $P_{\mathrm{b}} = P_{\mathrm{b}}^{\mathrm{T}} > 0$.

Next we show that $\overline{Q}$ is positive semi-definite for the given c. We rewrite $\overline{Q}$ in (11) as

$$\begin{aligned} \overline{Q} &= c\left[cL_{\mathrm{p}}^{\mathrm{T}} R_{\mathrm{b}} L_{\mathrm{p}} \otimes K^{\mathrm{T}} R_{\mathrm{a}} K - R_{\mathrm{b}} L_{\mathrm{p}} \otimes (A^{\mathrm{T}} P_{\mathrm{a}} + P_{\mathrm{a}} A^{\mathrm{T}})\right] \\ &= c\left[cL_{\mathrm{p}}^{\mathrm{T}} R_{\mathrm{b}} L_{\mathrm{p}} \otimes K^{\mathrm{T}} R_{\mathrm{a}} K + R_{\mathrm{b}} L_{\mathrm{p}} \otimes (Q_{\mathrm{a}} - K^{\mathrm{T}} R_{\mathrm{a}} K)\right] \\ &= c\left[\left(cL_{\mathrm{p}}^{\mathrm{T}} - I\right) R_{\mathrm{b}} L_{\mathrm{p}} \otimes K^{\mathrm{T}} R_{\mathrm{a}} K + R_{\mathrm{b}} L_{\mathrm{p}} \otimes Q_{\mathrm{a}}\right] \end{aligned} \tag{15}$$

Note that $R_{\mathrm{b}} L_{\mathrm{p}} \otimes Q_{\mathrm{a}} \geq 0$ since $Q_{\mathrm{a}} \geq 0$ and $R_{\mathrm{b}} L_{\mathrm{p}} > 0$ (this follows from (14)). Also $K^{\mathrm{T}} R_{\mathrm{a}} K \geq 0$. It follows that if $\left(cL_{\mathrm{p}}^{\mathrm{T}} - I\right) R_{\mathrm{b}} L_{\mathrm{p}} > 0$, then $\overline{Q} \geq 0$.

Consider the matrix $H = V^{\mathrm{T}}\left(cL_{\mathrm{p}}^{\mathrm{T}} - I\right) R_{\mathrm{b}} L_{\mathrm{p}} V$. Since V is nonsingular, it follows from [17, Obsv. 7.1.6] that if $H > 0$, then $\tilde{H} = (V^{-1})^{\mathrm{T}} V^{\mathrm{T}}\left(cL_{\mathrm{p}}^{\mathrm{T}} - I\right) R_{\mathrm{b}} L_{\mathrm{p}} V V^{-1} = \left(cL_{\mathrm{p}}^{\mathrm{T}} - I\right) R_{\mathrm{b}} L_{\mathrm{p}} > 0$. Thus, we will test positive semi-definiteness of the matrix $H = V^{\mathrm{T}}\left(cL_{\mathrm{p}}^{\mathrm{T}} - I\right) R_{\mathrm{b}} L_{\mathrm{p}} V$ as follows.

$$\begin{aligned} H &= V^{\mathrm{T}}\left(cL_{\mathrm{p}}^{\mathrm{T}} - I\right) R_{\mathrm{b}} L_{\mathrm{p}} V = cV^{\mathrm{T}} L_{\mathrm{p}}^{\mathrm{T}} R_{\mathrm{b}} L_{\mathrm{p}} V - V^{\mathrm{T}} R_{\mathrm{b}} L_{\mathrm{p}} V \\ &= cV^{\mathrm{T}} L_{\mathrm{p}}^{\mathrm{T}} (V^{-1})^{\mathrm{T}} V^{-1} L_{\mathrm{p}} V - V^{\mathrm{T}} (V^{-1})^{\mathrm{T}} V^{-1} L_{\mathrm{p}} V \\ &= c(V^{-1} L_{\mathrm{p}} V)^{\mathrm{T}} (V^{-1} L_{\mathrm{p}} V) - (V^{-1} V)^{\mathrm{T}} (V^{-1} L_{\mathrm{p}} V) \\ &= c\Lambda^2 - \Lambda = \Lambda(c\Lambda - I). \end{aligned} \tag{16}$$

The matrix $\Lambda = \mathrm{diag}[\lambda_1, \lambda_2, \ldots, \lambda_N]$ and recall that $0 < \lambda_{\min} \leq \lambda_i, \forall i$. It follows that $\Lambda > 0$ and if $(c\Lambda - I) > 0$, then also $\overline{Q} \geq 0$. We take $c = \frac{1}{\lambda_{\min}}$ to guarantee that $\overline{Q} \geq 0$.

We satisfied the conditions of [15, Thm. 2], so the control law $u = -cL_{\mathrm{p}} \otimes K$ is optimal with respect to (10). □

A. Discussion of optimality

The design procedure is as follows. First the local feedback matrix K is designed. For this only the agent model and the local weighting matrices $Q_{\mathrm{a}}, R_{\mathrm{a}}$ need to be known. This is done independently of the network. Such approach is very similar to those of [10], [11]. Then, when the cummunication topology is known and the weights in the Laplacian selected, the coupling gain c is set to satisfy $c \geq \frac{1}{\lambda_{\min}}$. If the Laplacian L_{p} is non-defective and has only real eingevaues, the control law is optimal with respect to (10). The feedback matrix K remains the same for all topologies. Note that although we require that the Laplacian has real eigenvalues, our approach is not limited to undirected graphs.

It is interesting to compare the optimal coupling gain c with the gain required to stabilize the system. In Theorem 1 the condition on optimal control is $c \geq \frac{1}{\lambda_{\min}}$, while for stability it is sufficient to take $c \geq \frac{1}{2\lambda_{\min}}$ [11]. Thus, for the graphs considered in Theorem 1, it suffices to increase the gains just twice to achieve optimality on top of stability.

Note that both matrices $\overline{R}$ and $\overline{Q}$ depend on the Kronecker product $R_{\mathrm{b}} \otimes R_{\mathrm{a}}$. Matrix $R_{\mathrm{b}} = (V^{-1})^{\mathrm{T}} V^{-1}$, from which follows that it has almost all entries non-zero. Consequently, matrix $\overline{Q}$ penalizes products of states of agent which are not neighbors. Similarly, matrix $\overline{R}$ does the same for inputs. In order to evaluate the criterion, all states must be known. This is kind of all-to-all coupling in the criterion. This might become important when the criterion is to be evaluated on-line, for instance in an MPC framework. However, the resulting control law is local and requires for each agent to know only the states of the neighboring agents.

This is in contrast to a more natural performance criterion with matrices $\overline{Q} = \mathrm{diag}[q_{11}, \ldots, q_{1n}, q_{21}, \ldots, q_{i1}, q_{i2}, \ldots, q_{nn}]$ and $\overline{R} = \mathrm{diag}[r_{11}, \ldots, r_{1m}, r_{21}, \ldots, r_{i1}, r_{i2}, \ldots, r_{mm}]$. Such criterion is local, because it needs only the states and inputs of individual agents. Nevertheless, it is well known that in general such control law results in a centralized control [8]. That is, all states of the multi-agent system have to be known by the ith agent to calculate its own control effort. Thus, we conjecture that all-to-all coupling must be present either in the criterion, or in the control law.

IV. Optimal control of vehicular platoons

In this section we will specialize the result on optimal control to vehicular platoons. We assume that there are N vehicles travelling in one-dimensional space. The vehicles are assumed to be identical and SISO. The model of the vehicle is given by (1), only the dimensions of matrices and vectors are different. They are $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}$ and $y_i \in \mathbb{R}$ is the position of the vehicle.

Only the nearest-neighbor interaction is allowed, hence the communication topology is a weighted path graph. The platoon is supposed to track the (virtual) leader which serves as a reference for the formation. The pinned Laplacian is

$$L_{\mathrm{p}} = \begin{bmatrix} p_1 + \epsilon_1 & -\epsilon_1 & 0 & \ldots & 0 \\ -1 & 1 + \epsilon_2 & -\epsilon_2 & 0 & \ldots \\ 0 & -1 & 1 + \epsilon_2 & -\epsilon_3 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & -1 & 1 + \epsilon_{N-1} & -\epsilon_{N-1} \\ 0 & \ldots & 0 & -1 & 1 \end{bmatrix} \tag{17}$$

The constant $\epsilon_i > 0$ is called a *constant of bidirectionality*. If $\epsilon_i = 1\,\forall i$ we have so called symmetric bidirectional control, if $\epsilon_i = 0\,\forall_i$ we have predecessor following. Otherwise we call the control law *asymmetric bidirectional control*.

The Laplacian is a non-symmetric tridiagonal matrix. Next we state some some useful properties of L_{p}.

Lemma 1. *Laplacian L_{p} in (17) and its eigenvalues λ_i have the following properties:*

a) *The eigenvalues λ_i are all real and $\lambda_i > 0\,\forall i$. The eigenvalues are ordered as $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$.*
b) *The eigenvalues are upper-bounded by $\lambda_{\max}$, that is, $\lambda_i \leq \lambda_{\max} \leq 2(1 + \epsilon_{\max})$ for $\epsilon_i \leq \epsilon_{\max}\,\forall i$.*
c) *Suppose that $\epsilon_i \leq \epsilon_{\max} < 1\,\forall i$. Then the eigenvalues $\lambda_1, \ldots, \lambda_N$ are lower-bounded by*

$$\lambda_i \geq \lambda_{\min} \geq \frac{1}{2}\frac{(1 - \epsilon_{\max})^2}{1 + \epsilon_{\max}} > 0, \qquad \forall i. \tag{18}$$

The bound is uniform, that is, it does not depend on N.

Proof: The property a) follows from the fact that L_{p} is a tridiagonal real matrix with non-positive off-diagonal terms, so its eigenvalues are real [18, Lem. 0.1.1]. b) The upper bound follows from Gershgorins theorem [17, Thm. 6.1.1], c) is proved in [5, Thm. 1]. □

The uniform boundedness of eigenvalues allows us to state a result about bounded distance of eigenvalues of the formation from the imaginary axis.

Lemma 2. *Suppose that the eigenvalues of L_{p} are real and uniformly bounded from zero and that the transfer function $M(s) = K(sI - A)^{-1}B$ has neither zero nor pole in the closed right half-plane, except for υ poles at the origin. Then, the eigenvalues ν_i of $I \otimes A - L_{\mathrm{p}} \otimes cBK$ in (6) are uniformly bounded from the origin for all N, that is,*

$$|\nu_i| \geq \xi > 0, \tag{19}$$

for some constant ξ depending on the open-loop model $M(s)$ and $\lambda_{\min}$.

Proof: It is well known [9] that the eigenvalues of any multi-agent system are given as a union of eigenvalues of

$$A - \lambda_i cBK. \tag{20}$$

Since our vehicle model is SISO and the Laplacian (17) has only real eigenvalues, the equation (20) is an equation of a standard root-locus plot [19] with a parameter λ_i. It is known that the root-locus curves start at the poles of the open loop and end at the zeros of the open loop. The open loop is in our case $M(s)$. When the parameter λ_i approaches zero, the roots of (20) will approach the eigenvalues of A.

By the assumption, the eigenvalues λ_i are uniformly bounded from below and above. Then, the eigenvalues of (20) cannot approach the eigenvalues of A, they must stay away from them for any formation size. Since the system has only poles at the origin and other poles and zeros are in the left half-plane, the eigenvalues of (20) are bounded away from the imaginary axis. Moreover, if the curves $A-\lambda_i cBK$ for $\lambda_i \in [\lambda_{\min}, \lambda_{\max}]$ are in the left half-plane, the system is stable. The distance ξ from the origin axis depends on the shape of the root-locus curves. □

As a special case for uniform boundedness of λ_i we have the Laplacian (17) with $\epsilon_i \leq \epsilon_{\max} < 1$. This results extend the result on uniform boundedness of the eigenvalues in [4], [20]. The most important consequence of the uniform boundedness is that the system will not get arbitrarily slow as the number of vehicles increases. This guarantees both controllability and reasonable convergence time.

Using the properties of the Laplacian, we can design an optimal control law for vehicular platoon.

Corollary 1. *Consider L_p in (17) with $0 < \epsilon_i \leq \epsilon_{\max} < 1$ for all N. Further let the matrices $Q_\mathrm{a} \geq 0, R_\mathrm{a} > 0$ be given and let K be given as in (8). Then the local state-feedback control law*

$$u = -c(L_\mathrm{p} \otimes K)x \tag{21}$$

with

$$c \geq \frac{2 + 2\epsilon_{\max}}{(1-\epsilon_{\max})^2} \tag{22}$$

is the optimal control law with respect to (10) for any N.

Proof: Recall the properties of the pinned Laplacian in Lemma 1: the eigenvalues are real, positive and bounded from below as $\lambda_i \geq \lambda_{\min} \geq \frac{(1-\epsilon_{\max})^2}{2+2\epsilon_{\max}}$ for all i. These properties satisfy the conditions in Theorem 1 and the result follows. □

The result holds for asymmetric platoons with stronger gains towards the leader $0 < \epsilon_i \leq \epsilon_{\max} < 1$. The corollary can be explained as follows. When a static state feedback matrix K is calculated, then it suffices to take a fixed gain c for any platoon size to achieve optimality. Hence, it is not necessary to increase the gain with the platoon size.

Remark 1. When a symmetric control is used ($\epsilon_i = 1 \, \forall i$), then the gain has to be increased with a quadratic rate with the number of vehicles growing. This is a consequence of the fact that the smallest eigenvalue of the Laplacian matrix decays to zero with a quadratic rate [21].

A. Scaling in asymmetric platoons

So far we have proved two positive effects of asymmetry $0 < \epsilon_i \leq \epsilon_{\max} < 1$ in vehicular platoons: the system has bounded eigenvalues from zero and also the state-feedback control is LQ optimal for any N. To this we might add other important quantity: the steady-state gain of any transfer function in the platoon is bounded [6, Thm. 1].

Therefore, using asymmetry might seems as a very good choice. However, for reasonable systems there is one important result which might prevent its use in vehicular platoons without centralized information. As was shown in [4]–[6], whenever the system has a uniform bound on eigenvalues and the norm of an agent's complementary sensitivity transfer function is greater than one, the $\mathcal{H}_\infty$ norm of any transfer function in platoon scales exponentially in N. Hence, large transient peaks are unavoidable.

The result [6, Thm. 2] was derived for output feedback. Here we will specialize it to state feedback considered in this paper. Suppose that there is an external input r_i acting at the agent with index i such that the input of its controller changes form (3) to $u_i = \left(\sum_{j=\mathcal{N}_i} cKl_{ij}(x_j - x_i)\right) + r_i$. This input might represent the desired distance or a measurement noise. We are interested in how this input affects the position of an agent with index j. Consider the transfer function

$$T_{ij}(s) = \frac{y_j(s)}{r_i(s)}. \tag{23}$$

Due to the interconnection in the formation, this transfer function is given by all vehicles in the platoon. Let $T_{\min}(s) = K(sI - A + c\lambda_{\min}BK)^{-1}B$ be the transfer function of a single agent, which use the feedback matrix also as its output matrix.

Theorem 2. *If $\|T_{\min}(s)\|_\infty > 1$ and the eigenvalues of L_p are uniformly bounded from zero, then for a transfer function in the platoon holds*

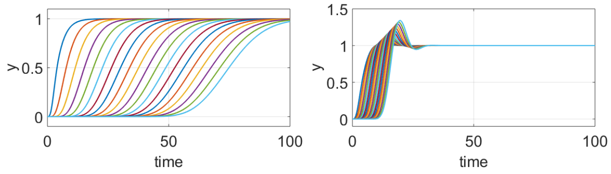
$$\|T_{ij}(s)\|_\infty \geq \zeta^{\delta_{ij}}\xi, \quad \zeta > 1, \xi > 0 \tag{24}$$

where δ_{ij} is the graph distance between the nodes i, j and ζ, ξ are constants independent of N.

Proof: This is a simple corollary of [6, Thm. 2]. Consider that the output matrix is K such that $\hat{y}_j = Kx_j$. Then the system would scale exactly as described in [6]. Since the interconnection between vehicles is realized via K, the only difference between our system and the system in [6] is the output equation of the jth agent. The output equation will not qualitatively change the exponential scaling proved in [6]. To see this, we write the transfer function as [22]

$$T_{ij}(s) = \frac{c(s)}{b(s)}\overline{T_{ij}}(s) \tag{25}$$

with $M(s) = \frac{c(s)}{a(s)} = C(sI - A)^{-1}B$, $\overline{M}(s) = \frac{b(s)}{a(s)} = K(sI - A)^{-1}B$ and $\overline{T_{ij}}(s)$ is the transfer function between r_i and $\hat{y}_j$. Let ω_0 be the frequency for which $|\overline{T_{ij}}(\jmath\omega)|$ attains its



(a) K_1. (b) K_2.

Fig. 1: Responses to step in leader's position for Σ_1, $N = 100, \epsilon_i = 0.5 \forall i$.

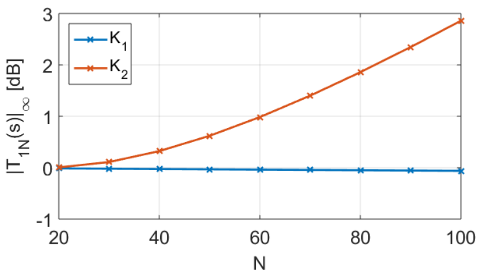
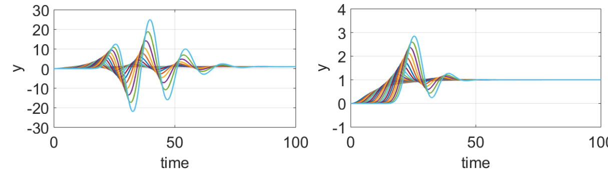


Fig. 2: $\|T_{1N}(s)\|_\infty$ as a function of N in semilogarithmic coordinates for Σ_1



(a) K_3. (b) K_4.

Fig. 3: Responses to step in leader's position for Σ_2, $N = 100, \epsilon_i = 0.5 \forall i$.

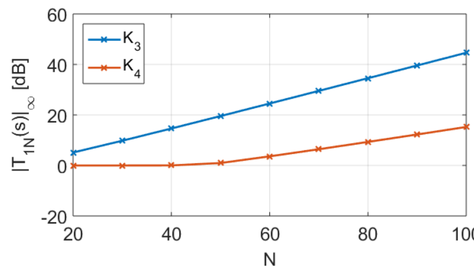


Fig. 4: $\|T_{1N}(s)\|_\infty$ as a function of N in semilogarithmic coordinates for Σ_2

maximum. For the system $\overline{T_{ij}}(s)$ it was shown in [6, Thm. 2] that $\|\overline{T_{ij}}(s)\|_\infty \geq \zeta^{\delta_{ij}} \xi_1^2 \overline{T_{ij}}(0)$. Taking $\xi = \xi_1^2 \overline{T_{ij}}(0) \left| \frac{c(\jmath\omega_0)}{b(\jmath\omega_0)} \right|$ gives the result (24). □

Despite the fact that we use optimal control, when the single agent closed-loop is not designed in a right way, exponential scaling occurs. There are cases when $\|T_{\min}(s)\|_\infty > 1$ is unavoidable. The most important of them is when there are at least two integrators in the open loop. It was shown in [1] that for such system $\|T_{\min}(s)\|_\infty > 1$. When there are in addition to that uniformly bounded eigenvalues of Laplacian, we must have exponential scaling. Hence, in any asymmetric control with $0 < \epsilon_i \leq \epsilon_{\max} < 1$ with two integrators in the open loop, exponential scaling occurs.

Two integrators in the open-loop are a necessary condition for tracking of the leader moving with constant velocity. This follows from the internal model principle [23]. Thus, exponential scaling is unavoidable even with optimal control. In other words, not even an optimal controller can overcome the inherent limitations given by the network structure. It can only mitigate the undesired effects.

In literature, the authors reported a positive effect of asymmetry even on the transients (e.g. [21]). This was thanks to the fact that the vehicles were allowed to know the leader's velocity, which is a centralized information. On the other hand, such vehicles can have only one integrator in the open loop. In this case the controller can be designed such that $\|T_{\min}(s)\|_\infty = 1$ and very good scaling and performance is achievable.

V. Simulations

In this section we verify the derived results using simulations. We will simulate two systems

$$\left.\begin{aligned} \dot{x}_i &= A_1 x_i + B u_i \\ y_i &= C x_i \end{aligned}\right\} := \Sigma_1, \quad \left.\begin{aligned} \dot{x}_i &= A_2 x_i + B u_i \\ y_i &= C x_i \end{aligned}\right\} := \Sigma_2. \tag{26}$$

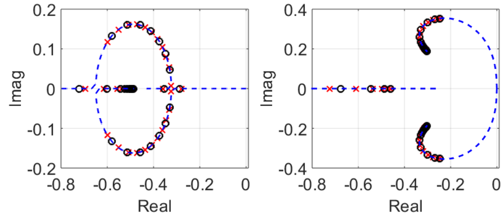
The matrices A_1 and A_2 are

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & -3 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -4 \end{bmatrix}, \tag{27}$$

$B = [0, 0, 0, 1]^{\mathrm{T}}$ and $C = [1, 0, 0, 0]$. The system Σ_1 has only one integrator in the open loop (Type-I), while Σ_2 is has two integrators in the open loop (Type-II). For each system and controller we used the same constant $\epsilon_i = 0.5 \, \forall i$. The coupling gain c was set using (22) to $c = 12$. The coupling weight with the leader was set to $p_1 = \frac{1}{cK_{i,1}}$ with $K_{i,1}$ being the first element of K_i. This was done in order to achieve unit steady-state gain from the leader's position to a position of all other vehicles (see [5, Cor. 1]).

First consider the system Σ_1. We use two pairs of weighting matrices $Q_{\mathrm{a}}, R_{\mathrm{a}}$. They are $Q_1 = \mathrm{diag}[0.5, 1, 1, 1]$, $R_1 = 10$ and $Q_2 = \mathrm{diag}[3, 1, 1, 1]$, $R_2 = 1$. The resulting feedback matrix for the first pair is $K_1 = [0.32, 0.85, 1.84, 1.67]$ and for the second pair it is $K_2 = [1.73, 3.67, 2.72, 1.23]$. The responses to the unit step in the leader's position are shown in Fig. 1. It is apparent that K_1 has slower response but has no overshoot. Scaling of $\mathcal{H}_\infty$ norms of the transfer function $T_{1N}(s)$ between the first and last vehicle for those two controllers is illustrated in Fig. 2. Clearly, the faster controller K_2 suffers from exponential scaling, while the norm of a system with K_1 remains constant. We can check that $\|T_{\min}(s)\|_\infty = 1.006$ for K_2, hence the conditions of Corollary 2 are satisfied and exponential scaling occurs. On the other hand, $\|T_{\min}(s)\|_\infty = 1$ for K_1 even with a scaled coupling gain $\lambda_{\max} c$, so condition [6, Thm. 3] is satisfied. This means that the $\mathcal{H}_\infty$ norm will be bounded in N. Thus, Type-I systems can have both good and bad scaling, despite the optimal controller.

For the system Σ_2 we also designed two controllers. The weighting matrices were $Q_3 = \mathrm{diag}[1, 1, 1, 1]$, $R_3 = 10$



(a) Σ_1, K_1. (b) Σ_2, K_3.

Fig. 5: Eigenvalues ν_i of $I \otimes A - L_\mathrm{p} \otimes cBK$ for $N = 40$ (black circle) and $N = 60$ (red crosses) and the corresponding root-locus curve $A - \lambda_i cBK$(dashed).

and $Q_4 = \mathrm{diag}[0.2, 15, 1, 0.1], R_4 = 10$. The resulting feedback matrices were $K_3 = [0.32, 1.18, 1.83, 1.66]$ and $K_4 = [0.14, 1.62, 3.00, 0.69]$. The step responses are shown in Fig. 3 and scaling of the $\mathcal{H}_\infty$ norm is in Fig. 4. It is clear that the system with K_4 has not only better transients, but also quantitatively slower scaling of the norm. Nevertheless, in both cases the norm increases exponentially in the number of vehicles. The conditions of Corollary 2 are satisfied in both cases: $\|T_{\min}(s)\|_\infty = 1.027$ for K_1 and $\|T_{\min}(s)\|_\infty = 1.024$ for K_2. This is a consequence of having two integrators in the open loop.

It is obvious that the transient with K_4 is much better than of K_3, although both controllers have similar norm of $\|K\| = \sqrt{K^\mathrm{T} K} \approx 3.5$. Therefore, for our setting is seems (also from simulations not shown here) that giving a lower weight to position and higher weight to velocity in the Q_a matrix helps the transient a lot. The transient for as much as 100 vehicles was acceptable and the design was easy.

The uniform bound on eigenvalues is illustrated in Fig. 5. The eigenvalues for both Σ_1 and Σ_2 do not get close to zero for any N.

VI. Conclusion

In this paper we presented a simple design of an LQR optimal control for a class of distributed system. The graph Laplacian must not be defective and must have only real eigenvalues. Then it is possible to design a controller which does not depend on the graph topology and change the gain afterwards based on the smallest eigenvalue of the Laplacian.

We applied this design approach to control of asymmetric vehicular platoons. Our control law guarantees optimal performance with a fixed gain for any formation size. We proved uniform bound on eigenvalues of the platoon. The transient performance was good even for 100 vehicles. However, it was also pointed out that there is an inherent limitation of asymmetric control: the $\mathcal{H}_\infty$ norm of a transfer function in the platoon grows exponentially in graph distance, when there are two integrators in the open loop. For a system with only one integrator in the open loop, the scaling depends on the controller.

An approach how to design a feedback matrix in order to obtain good scaling remains to be solved in future work. Moreover, the controller should be designed along with the weights ϵ_i in the Laplacian.

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