Nonzero bound on Fiedler eigenvalue causes exponential growth of H-infinity norm of vehicular platoon

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***Abstract*—We consider platoons composed of identical vehicles and controlled in a distributed way, that is, each vehicle has its own onboard controller. The regulation errors in spacing to the immediately preceeding and following vehicles are weighted differently by the onboard controller, which thus implements an asymmetric bidirectional control scheme. The weights can vary along the platoon. We prove that such platoons have a nonzero uniform bound on the second smallest eigenvalue of the graph Laplacian matrix—the *Fiedler eigenvalue*. Furthermore, it is shown that existence of this bound always signals undesirable scaling properties of the platoon. Namely, the $\mathcal{H}_\infty$ norm of the transfer function of the platoon grows exponentially with the number of vehicles regardless of the controllers used. Hence the benefits of a uniform gap in the spectrum of a Laplacian with an asymetric distributed controller are paid for by poor scaling as the number of vehicles grows.**

***Index Terms*—Vehicular platoons, Fiedler eigenvalue, harmonic instability, eigenvalues uniformly bounded from zero, asymmetric control, exponential scaling.**

I. Introduction

Platoons of vehicles offer promising solutions for future highway transport. They provide several advantages to the current highway traffic—they increase both the capacity and the safety of the highway and they allow the driver to relax.

Several platoon control architectures have been proposed in the literature. They differ mainly in presence of direct interactions with the platoon leader. If the information from the leader is permanently available to all following vehicles, the platoon can behave very well and is scalable. On the other hand, it requires establishing some communication among the vehicles, which can be disturbed or even denied by an intruder. Control schemes relying on communication comprise *leader following* and *cooperative adaptive cruise control*. For their overview and properties see e. g., [1]–[3].

Among the communication-free scenarios are the *predecessor following*, *constant time-headway spacing* and *bidirectional control* (*symmetric* or *asymmetric*). Recognizing their limitations, these architectures may still be useful as backup control solutions during communication failures. One of the key theoretical issues investigated with communication-free control schemes is *string stability*. Although there are variations among the concepts found in the literature (for a review see, e.g., [4]), the key idea is that the platoon is *string unstable* if the impact of a disturbance affecting one vehicle gets amplified as it propagates along the string (platoon). The predecessor-following strategy is string unstable if there are at least two integrators in the open loop of each vehicle [1]. Two integrators are a reasonable assumption, as they allow both velocity tracking and constant spacing [5]. The constant time-headway spacing policy increases the required intervehicular distances in response to the increased speed of the leader, which preserves the string stability [6]. For symmetric bidirectional formations, the response to noise (coherence) scales polynomially with the size of the platoon [7]. The paper also reveals the bad effect of increasing the number of integrators in the open loop.

Asymmetric controllers for platoons have received much attention after [8] was published. The authors show that for small controller asymmetry, the convergence rate of the least stable eigenvalue to zero (as the number of vehicles grows) decreases. Later the paper [9] shows that with a non-vanishingly small asymmetry, the least stable eigenvalue does not actually converge to zero but to some nonzero constant—a uniform nonzero lower bound can be achieved. This result guarantees a controllability of the formation consisting of an arbitrary number of vehicles. In [10] optimal localized control for asymmetric formation is proposed. The authors show that asymmetric control has beneficial effects on various performance measures. They do, however, assume that each vehicle in the platoon has the knowledge of the desired (leader's) velocity of the platoon. That information has to be communicated permanently to each vehicle by the leader. Our work to be presented differs in that we allow no communication among the vehicles.

The results in [11] reveal a significant drawback of the asymmetric control scheme. The paper analyzes a platoon of vehicles modeled by double integrators with a PD controller (equivalent to relative position and velocity feedback). They show that the peak in the magnitude frequency response of the position of the last vehicle to the change in the leader's position grows exponentially in the number of vehicles—a phenomenon labelled as *harmonic instability*. In contrast, if the controller is symmetric, the peak in the magnitude frequency response (the $\mathcal{H}_\infty$ system norm) only grows linearly [12]. Note that string instability merely means that the $\mathcal{H}_\infty$ norm is growing but harmonic instability means that it is growing very fast (in the number of vehicles).

With these results, several questions arise. Is harmonic instability present with any controller or can it be mitigated by some judicious choice of the controller structure? Can varying the asymmetry in the platoon counteract harmonic instability? Is harmonic instability an inherent property of an asymmetric control, or even of any uniformly bounded nearest neighbor interaction? In this paper we answer these questions.

We extend [11] to any open-loop model of a vehicle and any platoon with uniformly bounded eigenvalues. Our results also extend [1] from the predecessor-following architecture to any bidirectional asymmetric configuration. Moreover, we show that harmonic instability is, in fact, caused by the uniform boundedness and it is not possible to achieve a good scalability both in the convergence time (the bound on eigenvalues) and in the frequency domain (the $\mathcal{H}_\infty$ system norm). Some trade-off is necessary. This paper extends our previous conference

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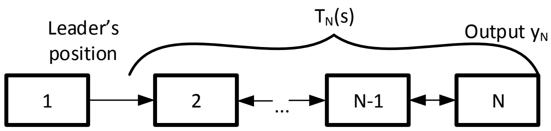


Fig. 1: Transfer function from the leader's position (second vehicle's input) to the trailing vehicle's output.

paper [13] to arbitrary asymmetric formations with controller gains and asymmetries varying among the vehicles.

The paper is structured as follows. First we give some preliminaries and provide definition of the harmonic instability. Then we prove uniform boundedness of a general platoon. In the next section the proof for the harmonic instability of an asymmetric control scheme is given. Finally some special cases are discussed and simulation results are shown.

II. Preliminaries and model

We assume N vehicles indexed by $i = 1, 2, \ldots, N$, travelling in a one-dimensional space. The first vehicle (indexed 1) is called *the leader* and it is controlled independently of the rest of the platoon. We analyze a bidirectional control, where each onboard controller measures the distances to its immediate predecessor and follower and strives to keep these close to the desired (reference) distance. It sets different weights to the front and rear regulation errors, hence *asymmetric bidirectional control*. We assume no intervehicular communication; all information is obtained only locally by the onboard sensors.

We study how the disturbance created by unexpected movements of the leader propagates along the platoon towards the final vehicle. Hence, we analyze the properties of the transfer function $T_N(s)$ from the leader's position to the position of the last vehicle as depicted in Fig. 1. Its frequency response and the way it scales with the number of vehicles N is used to prove harmonic instability for a given configuration.

Definition 1 (Harmonic stability [11])**.** Let $\gamma_N \equiv \sup_{\omega \in \mathbb{R}^+} |T_N(\jmath\omega)|$, where $\jmath = \sqrt{-1}$. The platoon is called *harmonically stable* if it is asymptotically stable and if $\limsup\limits_{N\to\infty} \gamma_N^{1/N} \leq 1$. Otherwise it is *harmonically unstable*.

An interpretation of harmonic instability is that some oscillatory motion of the leader has its amplitude magnified as it is propagated through the platoon and the growth of the magnitude is exponential in N [11]. In order words, the $\mathcal{H}_\infty$ norm of $T_N(s)$ grows exponentially with N.

Notation: We denote matrices by capital letters, vectors by lowercase letters and an element in a matrix A is denoted as a_{ij}. We use s as the complex variable in Laplace transform. The ith vector in a canonical basis is denoted $e_i \in \mathbb{R}^{N\times 1}$, that is, $e_i = [0, \ldots, 1, \ldots, 0]^T$, with 1 on the ith position. Identity matrix of size N is denoted as I_N.

A. System model

Each vehicle is described by an identical SISO transfer function $G(s) = \frac{b(s)}{a(s)}$. The output is the vehicle's position y_i. Dynamic controller described by a transfer function $R(s) = \frac{q(s)}{p(s)}$ is used to close the feedback loop. The input to the controller is defined in (2). The open-loop transfer function is $M(s) = R(s)G(s)$. From now on we will only use the open loop in the analysis. Its state-space description is

$$\dot{x}_i = Ax_i + Bu_i, \qquad y_i = Cx_i, \tag{1}$$

with $x_i \in \mathbb{R}^{n\times 1}$ as the state vector, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times 1}, C \in \mathbb{R}^{1\times n}$, and $y_i \in \mathbb{R}$ as the position of the vehicle.

The input has two parts $u_i = c_i + r_i$, where c_i is the part caused by coupling between vehicles and r_i external control signal, e.g., reference distances.

In platooning, the input u_i to each vehicle is a weighted sum of spacing errors to its predecessor in the string and its successor. Spacing error to the previous vehicle is weighted with $\mu_i > 0$ and the error to succeeding vehicle with $\mu_i\epsilon_i$. The asymmetry level $\epsilon_i \geq 0$ is therefore a ratio between front and rear gains. The weight μ_i as well as the asymmetry level ϵ_i can vary along the platoon. The input to the vehicle is then

$$u_i = \mu_i(y_{i-1} - y_i - d_{\text{ref}}) - \mu_i\epsilon_i(y_i - y_{i+1} - d_{\text{ref}}) \tag{2}$$

for $i = 2, \ldots, N$ and d_{ref} is a desired distance. The inter-vehicle coupling is then $c_i = \mu_i(y_{i-1} - y_i) - \mu_i\epsilon_i(y_i - y_{i+1})$ and the external input in this case is $r_i = \mu_i(-1 + \epsilon_i)d_{\text{ref}}$. In further development, we do not limit the external command r_i to be given only by the reference distance, it is treated as a general signal.

In a compact form, the stacked vector of inputs is $u = -Ly + r$, where L is a graph Laplacian representing an interconnection and has a form

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & 0 \\ -\mu_2 & \mu_2(1+\epsilon_2) & -\mu_2\epsilon_2 & 0 & \ldots & 0 \\ 0 & -\mu_3 & \mu_3(1+\epsilon_3) & -\mu_3\epsilon_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & -\mu_N & \mu_N \end{bmatrix}. \tag{3}$$

The vectors are $u = [u_1, \ldots, u_N]^T$, $y = [y_1, \ldots, y_N]^T$ and since the external input can be arbitrary, we take $r = [r_1, \ldots, r_N]^T$. The leader is controlled externally with the input u_1. The trailing vehicle has no follower and its input is $u_N = \mu_N(y_{N-1} - y_N - d_{\text{ref}})$.

Lemma 1. *The graph Laplacian in (3) has these properties*

a) $\lambda_1 = 0$ *is an eigenvalue of* L *with its eigenvector* $\mathbf{1} = [1, 1, \ldots, 1]^T$*, and this eigenvalue is simple,*
b) *all its eigenvalues are real, lie in the right-half plane, i.e.* $\lambda_i \geq 0$ *and are bounded by* $\lambda_i \leq \lambda_{\max} = 2\max(L_{ii})$.
c) L *can be partitioned as*

$$L = \left[\begin{array}{c|c} 0 & 0 \\ \hline \times & L_{\text{r}} \end{array}\right]. \tag{4}$$

and the spectrum of the reduced Laplacian L_{r} *coincides with all the nonzero eigenvalues of* L.

Proof. a) Simple zero eigenvalue follows from the presence of a directed spanning tree in the platoon. b) L is a tridiagonal real matrix with non-positive off-diagonal terms, so its eigenvalues are real [14, Lem. 0.1.1]. The upper bound follows from Geršgorin's theorem [15, Thm. 6.1.1]. c) Combining the property that the first row of L is zero and one of the eigenvalues is zero, similarity transformation reveals the eigenstructure described in the lemma. □

Using the last point, we can concentrate on the formation without the leader, because we removed the row corresponding to the leader and still kept all the nonzero eigenvalues. That's why the input to $T_N(s)$ acts at the second vehicle.

Definition 2 (Uniform boundedness)**.** The eigenvalues λ_i of a matrix $L \in \mathbb{R}^{N\times N}$ are uniformly bounded from zero if there exists a constant $\lambda_{\min} > 0$ such that $\lambda_i \geq \lambda_{\min}$ for $i = 2, \ldots, N$ and $\lambda_{\min}$ does not depend on N.

If the onboard controllers of all vehicles are asymmetric and have front gains stronger than the rear ones, then the uniform boundedness can be achieved. The proof of the following theorem is in the Appendix A.

Theorem 1. *If there is $\epsilon_{\max} < 1$ such that $\epsilon_i \leq \epsilon_{\max} \forall i$ and $\forall N$, then the nonzero eigenvalues of the Laplacian L given in (3) are uniformly bounded with $\lambda_{\min} \geq \frac{(1-\epsilon_{\max})^2}{2+2\epsilon_{\max}}$.*

B. Vehicle interconnection and diagonalization

Using a standard consensus or multi-vehicular formation notation [16], the overall formation model is

$$\dot{x} = \Big[I_N \otimes A - (I_N \otimes BC)(L \otimes I_n)\Big]x + (I_N \otimes B)r,$$
$$y = (I_N \otimes C)x, \tag{5}$$

where $x \in \mathbb{R}^{Nn\times 1}$ is a *stacked* state vector and $\otimes$ is the Kronecker product. We apply the approach of Theorem 1 from [16] but use Jordan instead of Schur decomposition. This will block diagonalize the system. The state transformation is $x = (V \otimes I_n)\hat{x}$, where $J = V^{-1}LV$ is the Jordan form of L. The matrix $V = [v_1, \ldots, v_N]$ is formed by (generalized) eigenvectors of L and v_{ji} is the jth element of the vector v_i. A block diagonal system is

$$\dot{\hat{x}} = [I_N \otimes A - J \otimes BC]\,\hat{x} + (V^{-1} \otimes B)r, \tag{6}$$
$$y = (V \otimes C)\hat{x}. \tag{7}$$

Consider a Jordan block in the block diagonal matrix (6). If it is of size one, it has the form

$$\dot{\hat{x}}_i = [A - \lambda_i BC]\,\hat{x}_i + Be_i^T V^{-1} r, \quad \hat{y}_i = C\hat{x}_i \tag{8}$$

This equation can be viewed as an output feedback system with a feedback gain λ_i and output $\hat{y}_i$. Its transfer function is

$$F_i(s) = \frac{M(s)}{1 + \lambda_i M(s)} = \frac{b(s)q(s)}{a(s)p(s) + \lambda_i b(s)q(s)}. \tag{9}$$

If the Jordan block has a size larger than one, it corresponds to identical blocks connected in series as in Fig. 2.

In the following we assume that all diagonal blocks are asymptotically stable for all N. As all the eigenvalues λ_i are real, design of a stable system is not a difficult task. We can use, e.g., the synchronization region approach [17] or the root-locus-like approach [18].

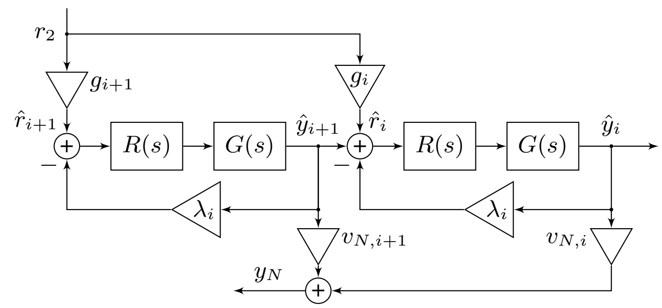


Fig. 2: Block diagram for a Jordan block of size two. The input is applied to the second vehicle and the output is the position of the Nth vehicle.

III. Harmonic instability

To test for harmonic instability, we examine the transfer function $T_N(s)$. The input to the platoon for such transfer function is $r(s) = [0, r_2(s), 0, \ldots, 0]^T = e_2 r_2(s)$. Based on (8), the input to the diagonal block $F_i(s)$ is the ith entry in the vector $\hat{r}$ given by $\hat{r}(s) = V^{-1}e_2 r_2(s) = g r_2(s)$ with $g = V^{-1}e_2$. The output of each block is (see Fig. 2)

$$\hat{y}_i(s) = F_i(s)\hat{r}_i(s) = F_i(s) g_i r_2(s). \tag{10}$$

The position y_N of the Nth vehicle can be calculated from (7). It is a weighted sum of the outputs of the blocks $\hat{y}$ with the weights equal to the Nth terms in the eigenvectors v_i

$$y_N(s) = \sum_{i=1}^{N} v_{Ni}\hat{y}_i(s) = \left[\sum_{i=1}^{N} v_{Ni} F_i(s) g_i\right] r_2(s), \tag{11}$$

with which we define $T_N(s) = \frac{y_N(s)}{r_2(s)} = \sum_{i=1}^{N} v_{Ni} F_i(s) g_i$.

The following product form of $T_N(s)$ holds for general platoons (both symmetric and asymmetric).

Theorem 2. *The transfer function from the input of the second vehicle to the position of the last vehicle in the system (5) with Laplacian (3) is given as*

$$T_N(s) = \frac{1}{\mu_2}\prod_{i=2}^{N} \lambda_i F_i(s) = \frac{1}{\mu_2}\prod_{i=2}^{N} T_{\lambda_i}(s), \tag{12}$$

We introduced $T_{\lambda_i}(s) = \lambda_i F_i(s)$ as a closed-loop transfer function with gain λ_i. The proof is given in the Appendix B.

Corollary 1. *For at least one integrator in the open loop $M(s)$, the steady-state gain of each block in (12) is $T_{\lambda_i}(0) = 1$, $\forall i$ and then $T_N(0) = 1$ if and only if $\mu_2 = 1$.*

Surprisingly, the greater the gain μ_2 (coupling with the leader), the lower the steady-state gain of the platoon.

Before stating the main theorem of the paper, we need to introduce the notation $M_{\min}(s) = \lambda_{\min} M(s)$ and define the closed-loop block for such open loop as

$$T_{\lambda_{\min}}(s) = \frac{M_{\min}(s)}{1 + M_{\min}(s)} = \frac{\lambda_{\min} b(s)q(s)}{a(s)p(s) + \lambda_{\min} b(s)q(s)}. \tag{13}$$

Theorem 3. *If the nonzero eigenvalues of Laplacian in (3) are uniformly bounded and $||T_{\lambda_{\min}}(s)||_\infty > 1$, then the platoon is harmonically unstable.*

Proof. The condition states that the closed-loop block $T_{\lambda_{\min}}(s)$ defined in (13) corresponding to the lower bound on the eigenvalues of Laplacian is greater in the $\mathcal{H}_\infty$ norm than one. Let ω_0 be the frequency at which the magnitude frequency response of this block attains its maximum. Further let $\alpha + \jmath\beta$ (with $\sqrt{-1} = \jmath$) be the value of the frequency response of the scaled open loop $M_{\min}$ at ω_0, i. e., $M_{\min}(\jmath\omega_0) = \alpha + \jmath\beta$. Then the squared modulus of the frequency response of the closed-loop $T_{\lambda_{\min}}(s)$ reads

$$|T_{\lambda_{\min}}(\jmath\omega_0)|^2 = \left|\frac{M_{\min}(\jmath\omega_0)}{1 + M_{\min}(\jmath\omega_0)}\right|^2 = \frac{\alpha^2 + \beta^2}{(\alpha+1)^2 + \beta^2}. \quad (14)$$

Since at ω_0 the closed-loop magnitude frequency response attains its maximum, the peak is greater than 1, i. e., $|T_{\lambda_{\min}}(0)| = 1 < |T_{\lambda_{\min}}(\jmath\omega_0)|$. From (14) we have

$$\frac{\alpha^2 + \beta^2}{(\alpha+1)^2 + \beta^2} > 1 \Rightarrow \alpha < -\frac{1}{2}. \quad (15)$$

The Laplacian eigenvalues can be ordered as $\lambda_{\min} \leq \lambda_2 < \ldots \leq \lambda_{\max}$. We can write $\lambda_i = \kappa_i \lambda_{\min}$ with gain $\kappa_i \in \langle 1, \frac{\lambda_{\max}}{\lambda_{\min}} \rangle$. By Lemma 1 all eigenvalues are real, so κ is real as well. By assumption in the theorem the bounds on κ do not depend on the number of vehicles.

Now the transfer function of each term in the product (12) is $T_{\lambda_i}(s) = \frac{\kappa_i M_{\min}(s)}{1+\kappa_i M_{\min}(s)}$ with $\kappa_i = \frac{\lambda_i}{\lambda_{\min}}$. We prove that all such transfer functions also have the magnitude frequency response at ω_0 greater than 1 (not necessarily their maximum there). The value of $M_{\min}(\jmath\omega_0)$ is still written as $\alpha + \jmath\beta$. The squared modulus of the closed-loop frequency response at ω_0 is

$$|T_{\lambda_i}(\jmath\omega_0)|^2 = \left|\frac{\kappa_i M_{\min}(\jmath\omega_0)}{1 + \kappa_i M_{\min}(\jmath\omega_0)}\right|^2 = 1 - \frac{2\kappa_i\alpha + 1}{(\kappa_i\alpha + 1)^2 + \kappa_i^2\beta^2}. \quad (16)$$

Since $\alpha < -\frac{1}{2}$, κ_i is real and greater than 1 and the denominator is positive, the sign of the fraction must be negative and (16) is greater than 1. Therefore, all transfer functions $T_{\lambda_i}(s)$ at ω_0 have the modulus greater than 1.

The modulus of the frequency response parametrized by κ attains its minimum at ω_0 for some κ_0, independent of the number of vehicles. This smallest modulus at ω_0 is denoted as $\zeta_{\min} > 1$ and it is unchanged for any and all diagonal blocks. By Theorem 2, the blocks are connected in series, therefore the total gain of the platoon is given by a product

$$|T_N(\jmath\omega_0)| = \prod_{i=2}^{N} |T_{\lambda_i}(\jmath\omega_0)| \geq (\zeta_{\min})^{N-1}. \quad (17)$$

The exponential growth of the peak in the magnitude frequency response has thus been proved. Although the eigenvalues of the Laplacian change upon adding more vehicles into the platoon, the bound on eigenvalues as well as the corresponding gain $\zeta_{\min}$ remain constant. □

To summarize, it suffices to test only a single transfer function $T_{\lambda_{\min}}(s)$ instead of the model of the whole platoon. If this transfer function is larger in $\mathcal{H}_\infty$ norm than one and there is a lower bound on the Fiedler's eigenvalue, the harmonic instability must occur and cannot be overcome by any linear controller. Note, however, that even systems with only one integrator in the open loop can be harmonically unstable.

IV. Special cases and simulations

A particularly important case is when there are two integrators in the open loop.

Lemma 2. *For at least two integrators in the open loop, frequency response of each term in the product (12) has a resonance peak, i. e.* $|T_{\lambda_i}(0)| = 1 < |T_{\lambda_i}(\jmath\omega_i)|$ *for some* ω_i.

Proof. Each term in the product in Theorem 2 is a closed-loop transfer function with at least two integrators in the open loop. For such system it was proved in Theorem 1 in [1] that it must have $\mathcal{H}_\infty$ norm greater than 1. □

Using the fact that $||T_{\lambda_{\min}}(s)||_\infty > 1$ with at least two integrators in the open loop, we satisfy the conditions in Theorem 3 and can extend the results of [1].

Corollary 2. *Vehicular platoon with uniformly bounded eigenvalues of Laplacian and at least two integrators in the open loop is harmonically unstable. This cannot be cancelled by any linear controller.*

Theorem 1 proves uniform bound for arbitrary asymmetric formation, so we can extend results of [11] to varying asymmetry and arbitrary dynamical models with two integrators.

Corollary 3. *Asymmetric bidirectional control with* $\epsilon_i \leq \epsilon_{\max} < 1 \,\forall i$, $\forall N$ *and with at least two integrators in the open loop is harmonically unstable.*

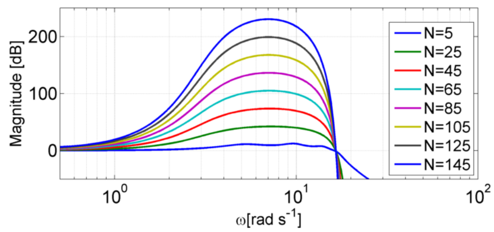
It was proved in [11], [18] that if the asymmetric platoon uses identical asymmetries $\epsilon_i = \epsilon$, $\mu_i = 1$, the eigenvalues of Laplacian are given in closed form as $\lambda_i = -2\sqrt{\epsilon}\cos\theta_i + 1 + \epsilon$, where θ_i is given as the ith solution of the nonlinear equation $\sin(N\theta_i) - \sqrt{\frac{1}{\epsilon}}\sin\left((N+1)\theta_i\right)$ on the interval $\langle 0, \pi \rangle$. The Laplacian eigenvalues are thus bounded and the bounds $\lambda_{\min} \geq (1-\sqrt{\epsilon})^2, \lambda_{\max} \leq (1+\sqrt{\epsilon})^2$ do not depend on N. Such formation satisfies the conditions of Corollary 3.

Another special case, which is harmonically unstable, is the predecessor following algorithm with $\epsilon_i = 0$, $\forall i$. On the other hand, harmonic stability of symmetric bidirectional control ($\epsilon_i = 1$) for double integrator model was proved in [12].

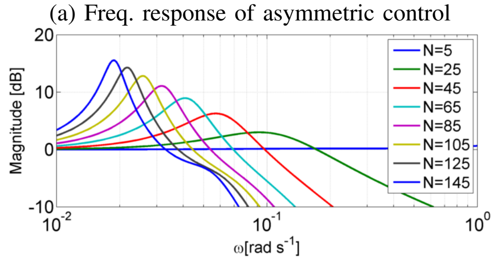
The simulation results comparing the asymmetric control with $\epsilon_i = 0.5$ and the symmetric control with $\epsilon_i = 1$ are shown in Fig. 3a and 3b. For the asymmetric control scheme it is apparent that as the number of vehicles grows, the peak in the magnitude frequency response grows exponentially and it is localized at almost identical and wide frequency range for any number of cars. The figure 3c shows step response of the platoon, which is oscillating and has very high overshot. The models used in all cases are $R(s) = \frac{110s^2+43s+3}{s^2+2.9s+1}$, $G(s) = \frac{1}{s^2}$. The controller has been chosen so that the overall system is asymptotically stable for any number of vehicles.

V. Conclusion

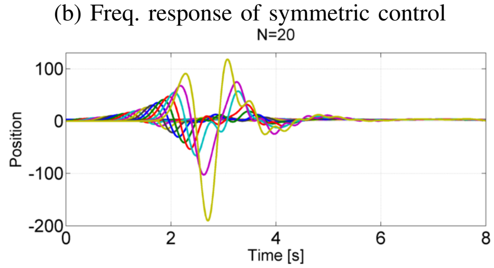
We dealt with a vehicular platoon controlled in a distributed and asymmetric way where each vehicle only measures the distance to its immediate neighbors. We studied *harmonic instability* of the platoon, which is a term for *exponential scaling* of the $\mathcal{H}_\infty$ norm of the transfer function of the platoon as the number of vehicles in the platoon grows.



(a) Freq. response of asymmetric control



(b) Freq. response of symmetric control



(c) Response to the leader's step in position.

Fig. 3: Figures 3a and 3b show frequency responses for a vehicular platoon with a growing number of vehicles. Figure 3c shows a response to leader's unit step change in position for an asymmetric platoon with 20 vehicles, $\epsilon = 0.5$.

The key condition for harmonic instability is the uniform boundedness of the Laplacian eigenvalues. For platoons with uniform boundedness we proposed a simple test consisting of evaluation of the $\mathcal{H}_\infty$ norm of a closed-loop transfer function of a single vehicle. The proof is based on a *product form* of the transfer function from the input of the second vehicle to the position of the last vehicle.

We proved uniform boundedness for a platoon with stronger front gains. In the case of two or more integrators in the open-loop transfer function and uniform bound on eigenvalues, harmonic instability cannot be overcome by any linear controller. The benefits of a uniform boundedness are thus paid for by a very bad scaling in the frequency response. Nonetheless, harmonic instability can also occur even in a situation with a single integrator in the open-loop model.

Appendix A
Proof of Theorem 1

Before we proceed to the proof, we state one useful Lemma.

Lemma 3. *[15, Cor. 6.1.6] Let $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ and let $p_1, \dots, p_n$ be positive numbers. Consider the matrix $B = P^{-1}AP$ with $P = \mathrm{diag}(p_1, \dots, p_n)$ and $b_{ij} = [p_j a_{ij}/p_i]$. Then all eigenvalues of A lie in the union of Geršgorin disks*

$$\bigcup_{i=1}^{n} \left\{ z \in \mathbb{C} : |z - a_{ii}| \le \frac{1}{p_i} \sum_{j=1, j\neq i}^{n} p_j |a_{ij}| \right\} \tag{18}$$

Proof of Theorem 1. With Lemma 3 we can get tighter bounds on λ_i by transforming the reduced Laplacian L_r into a *diagonally dominant* form $B = P^{-1}L_\mathrm{r}P$. After the transformation, each row of B reads

$$\left[\dots\ 0\ \ -\tfrac{p_{i-1}}{p_i}\mu_i\ \ \mu_i(1+\epsilon_i)\ \ -\tfrac{p_{i+1}}{p_i}\mu_i\epsilon_i\ \ 0\ \dots\right]. \tag{19}$$

To make it diagonally dominant, it must hold

$$-\frac{p_{i-1}}{p_i}\mu_i + \mu_i(1+\epsilon_i) - \frac{p_{i+1}}{p_i}\mu_i\epsilon_i \ge 0 \quad \forall i. \tag{20}$$

This is a difference inequality with variable p. We take p as

$$p = \frac{1}{2}\left(1 + \frac{1}{\epsilon_{\max}}\right), \tag{21}$$

which satisfies the inequality. Then P is a diagonal matrix $P = \mathrm{diag}(1, p, p^2, \dots, p^{N-2})$. Applying this transformation to L_r, we get the ith row

$$\left[\dots\ 0\ \ -\tfrac{1}{p}\mu_i\ \ \mu_i(1+\epsilon_i)\ \ -p\mu_i\epsilon_i\ \ 0\ \dots\right]. \tag{22}$$

The sum in each row equals the distance $d_i = \mu_i(1+\epsilon_i) - \frac{1}{p}\mu_i - p\mu_i\epsilon_i$ of Geršgorin's circle from zero and should be positive. After simple calculations, we obtain

$$d_i = \mu_i\left[-\frac{\epsilon_i}{2}\frac{1-\epsilon_{\max}}{\epsilon_{\max}} + \frac{1-\epsilon_{\max}}{1+\epsilon_{\max}}\right]. \tag{23}$$

Assume, without loss of generality, that $\mu_i \ge 1$. Then d_i in the equation above is minimized for $\epsilon_i = \epsilon_{\max}$. Therefore, the smallest distance of Geršgorin disks from zero, hence also the lower bound on the eigenvalues is

$$\lambda_{\min} \ge -\frac{1-\epsilon_{\max}}{2} + \frac{1-\epsilon_{\max}}{1+\epsilon_{\max}} = \frac{(1-\epsilon_{\max})^2}{2+2\epsilon_{\max}}. \tag{24}$$

Furthermore, it is positive for any $\epsilon_i \le \epsilon_{\max}$, making B diagonally dominant. To summarize, we found a bound which does not depend on the matrix size. □

Appendix B
Proof of Theorem 2

Before the proof, we need the following technical result.

Lemma 4. *Let $h_i = g_i\, v_{Ni}$. Then we have*

$$\sum_{i=2}^{N} h_i \lambda_i^m = 0 \textit{ for } m = 0, 1, \dots, N-3, \tag{25}$$

$$\sum_{i=2}^{N} h_i \frac{1}{\lambda_i} = \frac{1}{\mu_2}. \tag{26}$$

Proof. The terms in $h_i = g_i\, v_{Ni}$ can be written as $g_i = e_i^T V^{-1} e_2$ and $v_{Ni} = e_N^T V$. Plugging them into (25) yields

$$\sum_{i=2}^{N} h_i \lambda_i^m = e_N^T V J^m V^{-1} e_2 = e_N^T L^m e_2 = (L^m)_{N2} \tag{27}$$

where $(L^m)_{ij}$ is the (i,j) element of L^m. Laplacian is a banded matrix with nonzero diagonal and the first subdiagonal and by powering it, we add new nonzero bands. Hence, L can be powered at most $N-3$ times to keep zeros at

$(N-2)$th subdiagonal and the element $(L^m)_{N2} = 0$ for $m = 0, \ldots, N-3$.

Let $J_\mathrm{r} = V_\mathrm{r}^{-1} L_\mathrm{r} V_\mathrm{r}$ be the Jordan form of L_r. Equation (25) is obtained in a similar way using $V_\mathrm{r} J_\mathrm{r}^{-1} V_\mathrm{r}^{-1} = L_\mathrm{r}{}^{-1}$ as

$$\sum_{i=2}^{N} h_i \frac{1}{\lambda_i} = e_{N-1}^T V_\mathrm{r} J_\mathrm{r}^{-1} V_\mathrm{r}^{-1} e_1 = (L_\mathrm{r}{}^{-1})_{N-1,1} = \frac{1}{\mu_2}. \quad (28)$$

□

Proof of Theorem 2. For simplicity, only the case of *non-defective* Laplacian is shown here. First we need to evaluate a characteristic polynomial of L_r

$$\det(sI_{N-1}+L_\mathrm{r}) = s^{N-1}+\alpha_{N-2}s^{N-2}+\ldots+\alpha_1 s+\alpha_0, \quad (29)$$

where $\alpha_{N-2} = \mathrm{Tr}(L_\mathrm{r}) = \sum_{i=2}^{N} \lambda_i$ and $\alpha_0 = \prod_{i=2}^{N} \lambda_i$.

The transfer function $T_N(s)$ was defined in (11) as

$$T_N(s) = \sum_{i=1}^{N} g_i F_i(s) v_{N,i} = \sum_{i=1}^{N} g_i v_{N,i} \frac{b(s)q(s)}{a(s)p(s) + \lambda_i b(s)q(s)}. \quad (30)$$

Since $g_1 = 0$ (the leader cannot be controlled from the second vehicle), the block corresponding to $\lambda_1 = 0$ does not enter the sum (30), which then has $N-1$ terms and reads

$$T_N(s) = \frac{\sum_{i=2}^{N} h_i \psi \prod_{j=2, j\neq i}^{N} [\phi + \lambda_j \psi]}{\prod_{i=2}^{N} [\phi + \lambda_i \psi]}. \quad (31)$$

We define $\phi(s) = a(s)p(s)$, $\psi(s) = b(s)q(s)$ and $h_i = g_i v_{Ni}$. The argument s is omitted. The numerator of (31) is then

$$\begin{aligned} &\sum_{i=2}^{N} h_i \psi \prod_{j=2, j\neq i}^{N} [\phi + \lambda_j \psi] = \sum_{i=2}^{N} h_i \psi \Bigg\{ \phi^{N-2} \\ &+ \Bigg[\phi^{N-3}\psi \sum_{j=2, j\neq i}^{N} \lambda_j\Bigg] + \Bigg[\phi^{N-4}\psi^2 \sum_{j=2, k=2, j\neq k\neq i}^{N} \lambda_j \lambda_k\Bigg] \\ &+ \ldots + \Bigg[\phi^1 \psi^{N-3} \sum_{j=2, j\neq i}^{N} \prod_{k=2, k\neq i\neq j}^{N} \lambda_k\Bigg] + \Bigg[\psi^{N-2} \prod_{j=2, j\neq i}^{N} \lambda_j\Bigg] \Bigg\}. \end{aligned} \quad (32)$$

Let us put the terms with equal powers of $\phi^i \psi^j$ in (32) together. First, take those with $\phi^{N-2}\psi$. The sum $\phi^{N-2}\psi \sum_{i=2}^{N} h_i = 0$, using (25) in Lemma 4. Second, take those with $\phi^{N-3}\psi^2$:

$$\begin{aligned} &\phi^{N-3}\psi^2 \sum_{i=2}^{N} h_i \sum_{j=2, j\neq i}^{N} \lambda_j = \phi^{N-3}\psi^2 \sum_{i=2}^{N} h_i(\alpha_{N-2} - \lambda_i) \\ &= \phi^{N-3}\psi^2 \alpha_{N-2} \sum_{i=2}^{N} h_i - \phi^{N-3}\psi^2 \sum_{i=2}^{N} h_i \lambda_i = 0. \end{aligned} \quad (33)$$

We used the fact that $h_i(\lambda_2 + \ldots + \lambda_{i-1} + \lambda_{i+1} + \ldots + \lambda_N) = h_i(\alpha_{N-2} - \lambda_i)$ and then applied Lemma 4. Using similar constructions we arrive to the fact that all powers of $\phi^i \psi^j$ are multiplied by zero, so they vanish. The only exception is the term with ψ^{N-1}, for which we get

$$\psi^{N-1} \sum_{i=2}^{N} h_i \prod_{j=2, j\neq i}^{N} \lambda_i = \psi^{N-1} \alpha_0 \sum_{i=2}^{N} h_i \frac{1}{\lambda_i} = \psi^{N-1} \frac{\alpha_0}{\mu_2}. \quad (34)$$

We used the fact that $\prod_{i=2}^{N} \lambda_i = \alpha_0$, so $\prod_{j=2, j\neq i}^{N} \lambda_j = \frac{\alpha_0}{\lambda_i}$. The last equality follows from (26). With all these terms, the fraction in (31) simplifies to

$$\begin{aligned} T_N(s) &= \frac{1}{\mu_2} \frac{(b(s)q(s))^{N-1} \prod_{i=2}^{N} \lambda_i}{\prod_{i=2}^{N} [a(s)p(s) + \lambda_i b(s)q(s)]} \\ &= \frac{1}{\mu_2} \prod_{i=2}^{N} \frac{\lambda_i b(s)q(s)}{a(s)p(s) + \lambda_i b(s)q(s)}. \end{aligned} \quad (35)$$

This is a series interconnection of blocks $T_{\lambda_i}(s)$, which proves the theorem. The cases with defective Laplacian matrices can be treated in a similar way with all the results valid. □

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