Travelling waves in a multi-agent system with general graph topology

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Abstract: The paper presents a novel approach for the analysis of a multi-agent system with arbitrary interaction topology and identical agents. The approach is based on an irrational transfer function, the Wave transfer function, that decomposes the interactions between the agents and identifies the travelling waves in the multi-agent system. The approach based on the travelling waves describes the behaviour of the system from the local perspective, which is complementary to the traditional overall approach. The proposed approach allows us, for example, to describe the local effect of the agents that have more than two neighbours.

Keywords: multi-agent system, travelling wave, wave transfer function

1. INTRODUCTION

In recent years, the field of multi-agent systems is progressing rapidly mainly due to the current advances in mobile robotics, which has opened a large volume of possible applications. The multi-agent system can nowadays be represented, for instance, by a platoon of vehicles on a highway, a pack of mobile robots or more general systems such as social networks or even Google PageRank, see Chiang (2012).

The traditional approach for analysis of such a large-scale linear multi-agent system is by using the algebraic graph theory, see for instance works by Ren et al. (2007), Olfati-Saber et al. (2007) or Mesbahi and Egerstedt (2010) for an introduction to this problematic. Another approach is to diagonalize the Laplacian, see for instance work of Herman et al. (2014). The nonlinear systems are traditionally analysed either by the Lyapunov function, see Moreau (2005) and Zhang et al. (2012), or by examination of the passivity of the multi-agent system, see for instance Chopra and Spong (2006), Arcak (2007) or Zelazo and Mesbahi (2011).

These methods are very useful in determining whether the multi-agent system is (un)stable or reaches a consensus. However, they lack in describing the local behaviour of the system. For instance, it is very difficult to determine that some part of the multi-agent system is not properly tuned and amplifies/attenuates the signal, for instance a disturbance, or even causes oscillations. Recently, papers by Martinec et al. (2014a) and Martinec et al. (2014b) have examined so-called *Wave transfer functions* that are tailored for analysis of the local behaviour of a multi-agent system. These two papers describe the travelling waves in a path graph, where the agents are linear but non-identical. In this paper, we extend them by considering more general graph topologies, where the agents are identical but can have more than two neighbours. The paper provides a mathematical description of the reflection and transmission of the travelling wave and discusses some consequences of agent having more than two neighbours.

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2. MATHEMATICAL PRELIMINARIES

2.1 Mathematical model of agents

The behaviour of nth agent in a multi-agent system is described as

$$X_n(s) = P(s)U_n(s), \tag{1}$$

where s is the Laplace variable, $X_n(s)$ is an output of the agent, for instance a position, $P(s)$ is the transfer function of the agent, that means a model of the dynamics of the agent, and $U_n(s)$ is the input to the agent that is carried out by the local controller of the agent with the task to equalize $X_n(s)$ with outputs of its neighbouring agents. We assume that each agent may have arbitrary number of neighbours, hence,

$$U_n(s) = C(s) \sum_{k \in \mathcal{N}_b} (X_k(s) - X_n(s)), \tag{2}$$

where $\mathcal{N}_b$ is a set of neighbouring agents of the nth agent and $C(s)$ is the transfer function of the controller. The model of the agent is then

$$X_n(s) = M(s) \sum_{k \in \mathcal{N}_b} (X_k(s) - X_n(s)), \tag{3}$$

where $M(s) = P(s)C(s)$, which can be alternatively expressed as

$$X_n(s) = T_N(s) \sum_{k \in \mathcal{N}_b} X_k(s), \tag{4}$$

where $T_N(s) = M(s)/(1 + NM(s))$ and N is the number of neighbours of nth agent.

2.2 Travelling wave description

The key idea of the Wave transfer function approach for a multi-agent system with path graph topology, presented in Martinec et al. (2014a), is that we can represent the output of each agent, for instance the position, by two components, $A(s)$ and $B(s)$, which represent two waves propagating along the platoon in the forward and backward directions, respectively. This assumption is based on the standard argument for the solution of the wave equation, see for instance Asmar (2004) (d'Alembert's formula). The way how the wave propagates in the platoon is described by the *Wave transfer function* (WTF), which is an irrational transfer function defined described as

$$G(s) = \frac{X_{n+1}(s)}{X_n(s)} = \frac{1}{2}\alpha(s) - \frac{1}{2}\sqrt{\alpha^2(s) - 4}, \qquad (5)$$

if $N \to \infty$, where $\alpha(s) = 1/M(s)+2$ and $G^{-1}(s) = 1/G(s)$. The mathematical model of the platoon, see Fig. 1 is then given as

$$X_n(s) = A_n(s) + B_n(s), \qquad (6)$$
$$A_{n+1}(s) = G(s)A_n(s), \qquad (7)$$
$$B_n(s) = G(s)B_{n+1}(s), \qquad (8)$$

where $n \in \langle 1, N-1\rangle$. The leader, $n = 0$, and the rear-end agent, $n = N$, constitute boundaries and cause that the travelling wave reflects from them. These boundaries are called forced-end, described as

$$A_1(s) = G(s)X_0(s) - G^2(s)B_1(s), \qquad (9)$$

and free-end, described as

$$B_N(s) = G(s)A_N(s), \qquad (10)$$

respectively, where X_0 is the position of the externally-controlled leader.

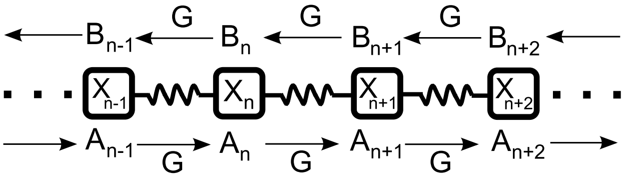

Fig. 1. Scheme of a multi-agent system with a path graph topology. The squares are agents with local dynamics described by $G(s)$. The virtual connections between the agents, which are created by the local control law, are illustrated by springs. All the agents are identical.

3. WAVES IN A MULTI-AGENT SYSTEM WITH OTHER THAN PATH GRAPH TOPOLOGY

This section shows how to generalize the description of the waves to a more general graph topologies. That is, to a topology where each agent is allowed to have more than two neighbours. Throughout the paper, we assume that the agents are identical.

3.1 Notation of the waves

To describe waves in a multi-agent system with more general graph topologies we need to introduce a different notation for the wave components. We replace A_n and B_n with $W^d_{n,n+1}$ and $W^a_{n+1,n}$, respectively. The first lower index is the index of the agent where the wave departs, while the second lower index is the index of the agent where the wave arrives. The upper index denotes if the wave is with the departing ('d') or arriving ('a') agent. The example is shown in Fig. 2.

Expressing (6) in this notation gives

$$\begin{aligned} X_n(s) &= W^d_{n,n+1}(s) + W^a_{n+1,n}(s) \\ &= W^a_{n-1,n}(s) + W^d_{n,n-1}(s), \end{aligned} \qquad (11)$$

where we assume that agents denoted $(n-1)$ and $(n+1)$ are neighbours of nth agent. Similarly, (7) and (8) are expressed as

$$W^a_{n,n+1}(s) = G(s)W^d_{n,n+1}(s), \qquad (12)$$
$$W^a_{n+1,n}(s) = G(s)W^d_{n+1,n}(s), \qquad (13)$$

respectively. Therefore, it holds for the example from Fig. 2 that $X_1 = W^d_{1,0} + W^a_{0,1} = W^d_{1,3} + W^a_{3,1} = W^d_{1,2} + W^a_{2,1}$ and $W^a_{1,2} = GW^d_{1,2}$.

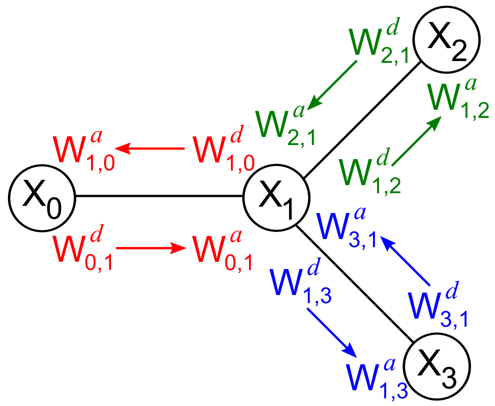

Fig. 2. Notation of the wave components in a multi-agent system.

3.2 Mathematical description of the waves

It was shown in Martinec et al. (2014a) that the wave travelling in a multi-agent system with a path graph topology and identical agents reflects only on the path-graph ends: the leader and the rear-end agent. The situation is more complicated for the case when the agent has more than two neighbours. In such a case, the wave partially reflects on this agent as well, which is described in the following theorem.

Theorem 1. The transfer function, $T_{t,N}(s)$, describing how the wave transmits through the agent with N neighbours, and the transfer function, $T_{r,N}(s)$, describing how the wave reflects from the agent with N neighbours, are

$$T_{t,N}(s) = \frac{W^d_{n,n-1}(s)}{W^a_{n+1,n}(s)} = \frac{T_N(s)(1 - G^2(s))}{G(s)(1 - NT_N(s)G(s))}, \; N \geq 2 \qquad (14)$$

$$\begin{aligned} T_{r,N}(s) &= \frac{W^d_{n,n-1}(s)}{W^a_{n-1,n}(s)} \\ &= \frac{(N-1)T_N(s)G^2(s) + T_N(s) - G(s)}{G(s)(1 - NT_N(s)G(s))}, \; N \geq 1, \end{aligned} \qquad (15)$$

respectively, where $(n-1)$th and $(n+1)$th are neighbouring agents of nth agent.

Proof. We substitute (11) into (4) to obtain

$$W^a_{n-1,n} + W^d_{n,n-1} = T_N \sum_{k \in \mathcal{N}_b} (W^d_{k,n} + W^a_{n,k}), \qquad (16)$$

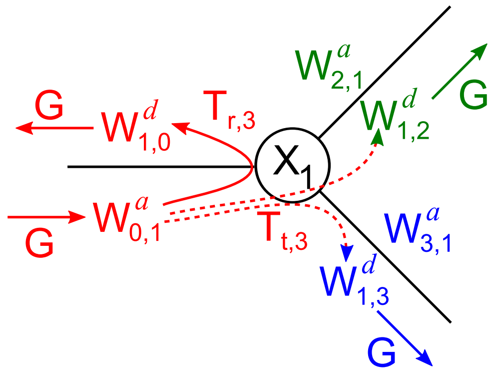

Fig. 3. The detail of the reflection and transmissions of the wave travelling from the 0th agent from Fig. 2, where G is the Wave transfer function from (5), $T_{\mathrm{t},3}$ and $T_{\mathrm{r},3}$ are the transfer functions defined in (14) and (15), respectively, with $N = 3$. Notice that all agents are identical. The waves travelling from 2nd or 3th agent are propagated analogously.

where we assumed that $(n-1)$th agent is the neighbour of nth agent, hence $(n-1)$th agent is also a part of set $\mathcal{N}_b$. Eq. (11) can be alternatively expressed as

$$W^a_{n,k} = GX_n - GW^a_{k,n} = GW^a_{n-1,n} + GW^d_{n,n-1} - GW^a_{k,n}. \tag{17}$$

Substituting (17) into (16) and rearranging it gives

$$W^a_{n-1,n} + W^d_{n,n-1} = T_N(G^{-1} - G) \sum_{k\in\mathcal{N}_b} (W^a_{k,n}) + NT_N G(W^a_{n-1,n} + W^d_{n,n-1}). \tag{18}$$

Next, we separate the wave arriving from $(n-1)$th agent, that is $W^a_{n-1,n}$, which yields

$$(1-NT_N G)W^d_{n,n-1} = T_N(G^{-1} - G) \sum_{k\in\mathcal{N}_b, k\neq(n-1)} (W^a_{k,n}) + (NT_N G + T_N(G^{-1} - G) - 1)W^a_{n-1,n}. \tag{19}$$

Finally, separating the wave departing from nth agent to $(n-1)$th agent, $W^d_{n,n-1}$, gives

$$W^d_{n,n-1} = \frac{T_N(1-G^2)}{G(1-NT_N G)} \sum_{k\in\mathcal{N}_b, k\neq(n-1)} W^a_{k,n} + \frac{(N-1)T_N G^2 + T_N - G}{G(1-NT_N G)} W^a_{n-1,n}, \tag{20}$$

where

$$T_{\mathrm{r},N} = \frac{W^d_{n,n-1}}{W^a_{n-1,n}} = \frac{(N-1)T_N G^2 + T_N - G}{G(1-NT_N G)} \tag{21}$$

is the transfer function which describes reflection of the wave from the agent and

$$T_{\mathrm{t},N} = \frac{W^d_{n,n-1}}{W^a_{k,n}} = \frac{T_N(1-G^2)}{G(1-NT_N G)} \tag{22}$$

describes transmission of the wave through the agent. □

The interpretation of the theorem is as follows. If there is a wave travelling to the agent with more than two neighbours, then it is partially reflected from this agent (described by $T_{\mathrm{r},N}(s)$) and partially transmitted (described by $T_{\mathrm{t},N}(s)$). For example, the output $X_1(s)$ of the multi-agent system from Fig. 3 can be expressed as

$$X_1 = (1 + T_{\mathrm{r},3})W^a_{0,1} + T_{\mathrm{t},3}W^a_{2,1} + T_{\mathrm{t},3}W^a_{3,1}. \tag{23}$$

The output is composed of three parts, since the agent has three neighbours,: i) wave that travels from 0th agent and reflects back to 0th agent, ii) wave that travels from 2nd agent and transmits to the 0th agent, and iii) wave that travels from 3rd agent and transmits to the 0th agent. We note that we can equivalently express $X_1(s)$ also by calculating the reflected wave travelling either from 2nd or 3rd agent.

3.3 Properties of the waves

Theorem 1 reveals interesting relation between the transmitted and reflected waves. It is described by the following corollary.

Corollary 2. The transfer functions $T_{\mathrm{t},N}$ and $T_{\mathrm{r},N}$ are related as follows

$$T_{\mathrm{t},N} - T_{\mathrm{r},N} = 1, \tag{24}$$

for $N \geq 2$.

Proof. By straightforward substitution for $T_{\mathrm{t},N}$ and $T_{\mathrm{r},N}$ from Theorem 1. □

Theorem 1 is in agreement with the result from Martinec et al. (2014a) since, by the following corollary, the wave in a multi-agent system with a path graph topology and identical agents reflects only on the path-graph ends.

Corollary 3. If the agent in a multi-agent system has exactly two neighbours and if the agents are identical then the wave does not reflect from this agent.

Proof. Substituting $N = 2$ into (15) gives

$$T_{\mathrm{r},2} = \frac{T_2 G^2 + T_2 - G}{G - 2T_2 G^2} = \frac{T_2 G + T_2 G^{-1} - 1}{1 - 2T_2 G}. \tag{25}$$

We substitute $T_2 = 1/\alpha$ and for G from (5) and get

$$T_{\mathrm{r},2} = \frac{\alpha^{-1}\left(\frac{1}{2}\alpha - \frac{1}{2}\sqrt{\alpha^2-4}\right) + \alpha^{-1}\left(\frac{1}{2}\alpha + \frac{1}{2}\sqrt{\alpha^2-4}\right) - 1}{1 - \frac{2}{\alpha}\left(\alpha\frac{1}{2} - \frac{1}{2}\sqrt{\alpha^2-4}\right)} = \frac{0}{\alpha^{-1}\sqrt{\alpha^2-4}} = 0, \tag{26}$$

where we used $G^{-1} = \alpha/2 + \left(\sqrt{\alpha^2-4}\right)/2$. Since $T_{\mathrm{r},2} = 0$, the wave does not reflect from the agent. □

One of the important properties of the transfer function is the DC gain. The DC gain can be used to analyse the amplification/attenuation of the travelling wave as it propagates through the agent. The simulation example is shown in Section 4.2.

Lemma 4. If the open-loop model of the agent, $M(s)$, has at least one integrator, then the DC gains of $T_{\mathrm{t},N}$ and $T_{\mathrm{r},N}$ depend only on the number of neighbours, N. Specifically,

$$\kappa_{\mathrm{t}} = \lim_{s\to 0} T_{\mathrm{t},N}(s) = \frac{2}{N}, \tag{27}$$

$$\kappa_{\mathrm{r}} = \lim_{s\to 0} T_{\mathrm{r},N}(s) = \frac{2}{N} - 1, \tag{28}$$

where κ_{t} and κ_{r} are the DC gains of the transfer functions $T_{\mathrm{t},N}$ and $T_{\mathrm{r},N}$, respectively.

Proof. The proof is given in Appendix A. □

4. MATHEMATICAL SIMULATIONS

4.1 The travelling waves in the multi-agent system

The mathematical simulations are carried out for $P(s) = 1/(s(s+4))$ and $C(s) = (4s+4)/s$, which represent a second order system with a linear friction controlled by a PI controller. Hence, the overall model of the agent is $M(s) = (4s+4)/(s^2(s+4))$. The topology of the multi-agent system that we assume is in Fig. 4. We can see that only the agent indexed P_7 has three neighbours. We choose such a graph topology on purpose to make the wave transmissions and reflections more transparent.

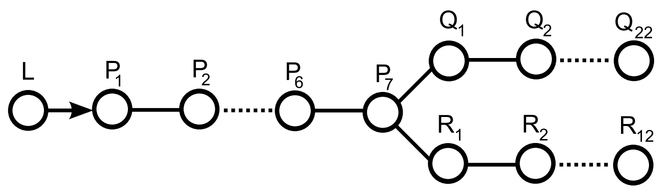

Fig. 4. The topology of the multi-agent system used for simulations, where the 'L-node' is the externally controlled leader of the system.

The way how the wave propagates in the system with the topology in Fig. 4 is shown in Fig. 5. The individual panels show:

top-left The wave is initiated by the leader and propagates to agent P_7.
top-middle The wave is being transmitted to 'Q' and 'R' parts of the graph (blue circles) and it is also reflected back to 'P' part (red crosses).
top-right The wave is reflected from agent R_{12} and propagates back to agent P_7 (red crosses).
bottom-left and middle The wave arriving to agent P_7 from the 'R' part is transmitted to 'P' and 'Q' parts of the graph. We can see it by a rise of $W^a_{i+1,i}$ (red crosses) for P_1 - P_7 and $W^a_{i-1,i}$ (blue circles).
bottom-right The wave travelling from P_7 to P_1 is reflected from the leader with the negative sign and travels back to P_7. We can see it by a drop of $W^a_{i-1,i}$ for P_1 - P_7 (blue circles).

Independent numerical validation of Theorem 1 is shown in top panel of Fig. 6. We can see excellent agreement between the state-space and the 'wave' approach. The bottom panel shows individual waves traveling through the agent P_7, where $W1 = W^a_{P6,P7}$, $W2 = T_{\mathrm{r},3} W^a_{P6,P7}$, $W3 = T_{\mathrm{t},3} W^a_{Q1,P7}$ and $W4 = T_{\mathrm{t},3} W^a_{R1,P7}$. In other words, $W2$ is the wave that travels from P part of the graph and reflects from agent P_7, and $W3$ and $W4$ are waves that travel from Q and R parts of the graph, respectively, and transmit through agent P_7. Therefore,

$$W^d_{P7,P6} = W2 + W3 + W4 \tag{29}$$

and

$$\begin{aligned} X_{P7} &= W^a_{P6,P7} + W^d_{P7,P6} \\ &= W^a_{P6,P7} + T_{\mathrm{r},3} W^a_{P6,P7} + T_{\mathrm{t},3} W^a_{Q1,P7} + T_{\mathrm{t},3} W^a_{R1,P7}. \end{aligned} \tag{30}$$

We note that although the numerical validation is done for a system with acyclic graph topology, the travelling wave

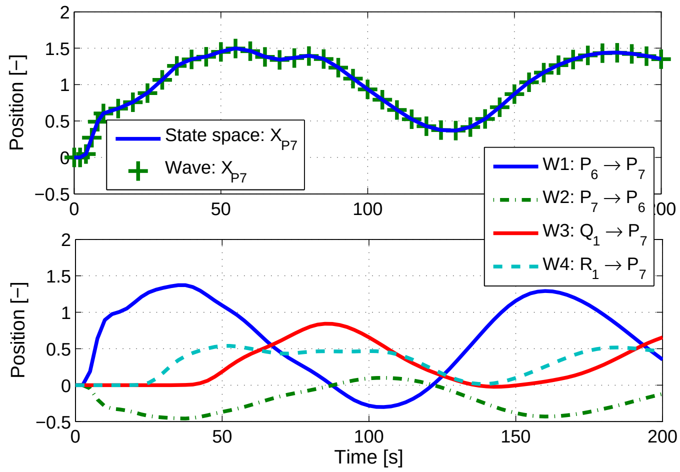

Fig. 6. The top panel shows the comparison of positions of the agent denoted as P_7 from Fig. 5 simulated by the state-space approach using (4) (solid line) and that computed by 'wave' approach using (14) and (15) (plus signs). The bottom panel shows individual contributions of three waves arriving from the neighbouring agents of agent P_7.

approach is applicable also for a system with topology that contains cycles.

4.2 The effect of many neighbours

The local effect of the agent with more than two neighbours is demonstrated on the topology shown in Fig. 7. The response of the system, when the leader changes its position from 0 to 1, is given in Fig. 8. We can see that the more neighbours the agent has the smaller amplitude is propagated through the agent. The figure also numerically verifies Lemma 4. Although the DC gain determines the steady state of the system, we can use it to approximate its output, in this case the position, even before the whole multi-agent system reaches the steady state. The transmitted wave is almost settled for time between 35 and 45 seconds and the reflected wave returns back to the P_2 agent after about 45 seconds. Therefore the position of the P_2 agent between 35 and 45 seconds is approximated by (27).

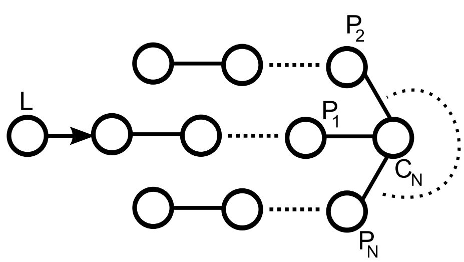

Fig. 7. The star-graph topology of the multi-agent system. The central agent, $\mathrm{C_N}$, has N neighbours. Each branch of the star graph has 20 agents with 'L-node' being the leader of the system.

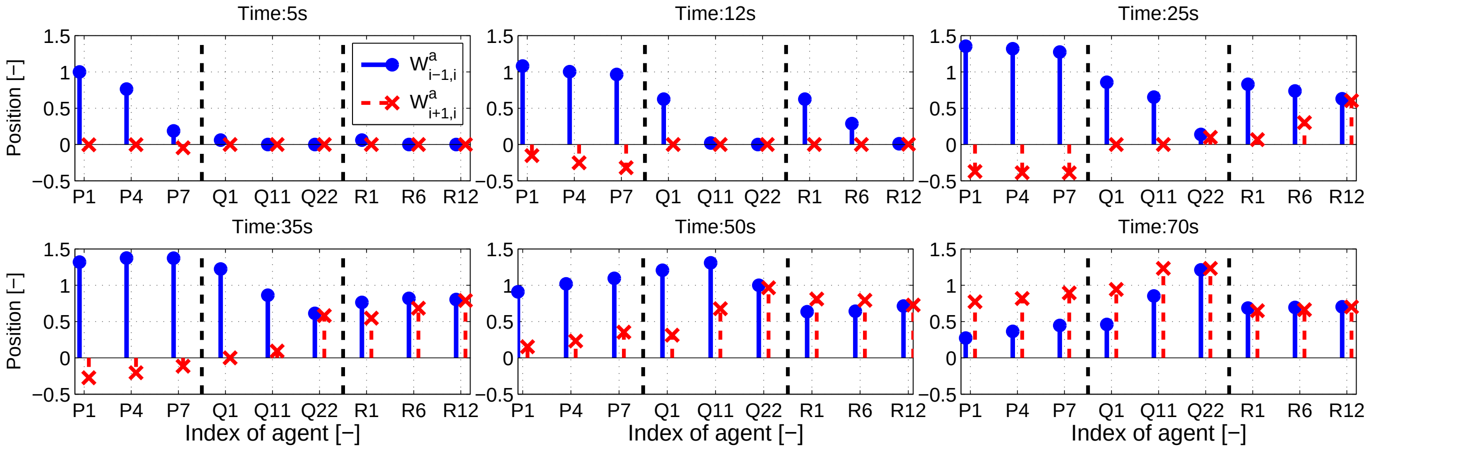

Fig. 5. The mathematical simulation of the wave propagating in the graph from Fig. 4. At the beginning, $t = 0\,\mathrm{s}$, all agents are at position 0 except of leader, which changes position from 0 to 1. At intermediate times, the wave travels to agent P_7, where it is partially transmitted to the 'Q' and 'R' parts of the graph and partially reflected back to the leader. The blue circles and red crosses represent $W^a_{i-1,i}$ and $W^a_{i+1,i}$ components of the wave on the ith agent, respectively. We can imagine these components as the left-to-right and right-to-left wave in the graph from Fig. 4, or as A and B components in the notation from (6), respectively.

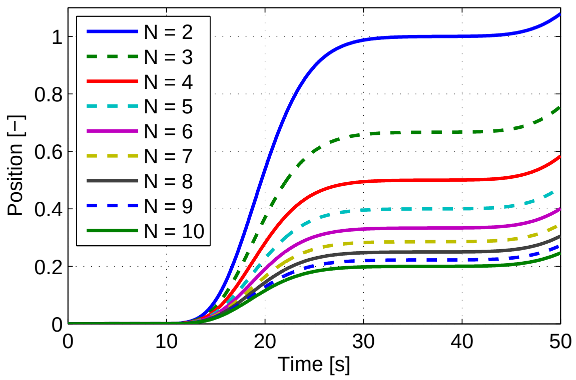

Fig. 8. Mathematical simulations of multi-agent systems with different interaction topology. The topology is a star graph, see Fig 7, with N denoting the number of neighbours of the central agent $\mathrm{C_N}$. The vertical axis shows position of agent $\mathrm{P_2}$.

5. DISCUSSION

The traditional approach based on the state-space description of the multi-agent system allows us to find a closed form formula for the transfer function among the outputs of the agents. For example, we can find the transfer function $T_{P7,Q1}(s)$ from the output of P_7th agent to the output of Q_1th agent in Fig. 4. This transfer function $T_{P7,Q1}(s)$ then captures all the interactions among the agents and effect of the boundary conditions of the system. Hence the transfer function describes the overall behaviour of the multi-agent system. This overall description is very useful in determining whether the multi-agent system is (un)stable or reaches a consensus. However, it lacks in describing the local behaviour of the system.

On the other hand, the transfer function $W^a_{P7,Q1}/W^a_{P6,P7}$ or $W^a_{P7,Q1}/W^a_{R1,P7}$ describes the interaction among the agents from the local perspective, which means that it does not consider neither the interactions among other agents nor the effect of the boundary conditions. The closed loop formulas for this local interaction are given in Theorem 1. The approximate amplitude of the transmitted/reflected wave is given by the DC gains carried out in Lemma 4 and shown in Fig. 8. This local, travelling-wave, description is the main contribution of the paper.

The travelling wave approach can also be used to describe the overall behaviour of the system as shown in Fig. 6. However, this again requires to consider interactions among all the agents, which is rather cumbersome with this approach.

Therefore, we consider the travelling wave approach as a complementary approach to the traditional state-space approach, since it gives insight into local interaction of a multi-agent system, which is particularly useful for a large-scale multi-agent system.

6. CONCLUSIONS

This paper describes the behaviour of the multi-agent system from the local point of view. It reveals the wave-like behaviour, which is particularly apparent in a large-scale multi-agent system. The main advantage of the proposed description is that it analyses the local behaviour of individual agents. By the analysis of the travelling wave and its reflection on the agents, we can estimate, how the output, for instance the position, of one agent propagates in the multi-agent system. This can serve, for example, to identify the locations of the agents that are responsible for unnecessary amplification/attenuation of the wave.

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Appendix A. PROOF OF LEMMA 4

If there is at least one integrator in the open-loop system of the agent, $M(s)$, then

$$\lim_{s\to 0} T_N = \lim_{s\to 0} \frac{M(s)}{1+NM(s)} = \frac{1}{N} \tag{A.1}$$

and

$$\lim_{s\to 0} G = \lim_{s\to 0} \frac{1}{2}\alpha(s) - \frac{1}{2}\sqrt{\alpha^2(s)-4} = 1, \tag{A.2}$$

since $\lim_{s\to 0}\alpha(s) = \lim_{s\to 0} 1/M(s) + 2 = 2$. Then

$$\kappa_\mathrm{t} = \lim_{s\to 0} T_{\mathrm{t},N} = \lim_{s\to 0}\frac{1}{G^2}\lim_{s\to 0}\frac{T_N(1-G^2)}{G^{-1}-NT_N} = 1\cdot\frac{0}{0}. \tag{A.3}$$

Applying L'Hopital's rule on the second limit gives

$$\begin{aligned} T_{\mathrm{lim}} &= \lim_{s\to 0}\frac{T_N'(1-G^2)+T_N(-2GG')}{-G^{-2}G'-NT_N'} \\ &= \lim_{s\to 0}\frac{\dfrac{T_N'(1-G^2)}{G'}+T_N(-2G)}{-G^{-2}-\dfrac{NT_N'}{G'}} \end{aligned} \tag{A.4}$$

where $'$ denotes derivative with respect to the Laplace variable s.

We denote $M(s) = n(s)/d(s)$, where $n(s)$ and $d(s)$ are the numerator and denominator polynomials of $M(s)$, respectively. Then,

$$T_N'(s) = \left(\frac{n(s)}{d(s)+Nn(s)}\right)' = \frac{n'(s)d(s)-n(s)d'(s)}{(d(s)+Nn(s))^2}. \tag{A.5}$$

and

$$G' = \frac{1}{2}\alpha' - \frac{\alpha\alpha'}{2\sqrt{\alpha^2-4}}, \tag{A.6}$$

where

$$\alpha' = \frac{d'(s)n(s)-d(s)n'(s)}{n^2(s)}. \tag{A.7}$$

Now, we need to distinguish between the number of integrators in $M(s)$.

A.1 One integrator in $M(s)$

One integrator in $M(s)$ causes that $\lim_{s\to 0} d'(s) \neq 0$ and $\lim_{s\to 0} d(s) = 0$, hence $\lim_{s\to 0} T_N' \neq 0$ and $|\lim_{s\to 0} T_N'| \leq \infty$.

Due to one integrator also $\lim_{s\to 0}\alpha' \neq 0$ and consequently $\lim_{s\to 0} G' = -\infty$. Substituting for T_N, T_N', G and G' into (A.4) gives $T_{\mathrm{lim}} = 2/N$ then

$$\kappa_\mathrm{t} = \lim_{s\to 0} T_{\mathrm{t},N} = \lim_{s\to 0}\frac{1}{G^2}T_{\mathrm{lim}} = 1\cdot\frac{2}{N} = \frac{2}{N}. \tag{A.8}$$

A.2 More than one integrator in the OL

For two and more integrators is $\lim_{s\to 0} d'(s) = 0$ and $\lim_{s\to 0} T_N' = 0$, but $\lim_{s\to 0}\alpha' = 0$. Hence, we need to separately treat limit $\alpha'/\sqrt{\alpha^2-4}$ as follows

$$\begin{aligned} \frac{\alpha'}{\sqrt{\alpha^2-4}} &= \frac{d'(s)n(s)-d(s)n'(s)}{n(s)\sqrt{d^2(s)+4d(s)n(s)}} = \\ &= \frac{s^{p/2}\left(s^{(p/2)-1}\overline{d'}(s)n(s) - s^{p/2}\overline{d}(s)n'(s)\right)}{s^{p/2}n(s)\sqrt{s^p\overline{d^2}(s)+4\overline{d}(s)n(s)}}, \end{aligned} \tag{A.9}$$

where p is the number of integrators in $M(s)$ and the overline symbol denotes the polynomial that has factored out the highest power of s, e.g. $d(s) = s^p\overline{d}(s)$ or $d'(s) = s^{(p-1)}\overline{d'}(s)$. Carrying out the limit of G' in view of (A.9) gives

$$\lim_{s\to 0} G'(s) = \begin{cases} -\infty & \text{if } p = 1, \\ \lambda\in\mathbb{R} & \text{if } p = 2, \\ 0 & \text{if } p \geq 3. \end{cases} \tag{A.10}$$

Therefore, substituting for T_N', G and G' into (A.4) gives

$$\kappa_\mathrm{t} = \lim_{s\to 0} T_{\mathrm{t},N} = \lim_{s\to 0}\frac{1}{G^2}T_{\mathrm{lim}} = 1\cdot 2\lim_{s\to 0} T_N = \frac{2}{N}. \tag{A.11}$$

The DC gain of the 'reflected' transfer function is carried out from (24) as

$$\kappa_\mathrm{r} = \kappa_\mathrm{t} - 1. \tag{A.12}$$