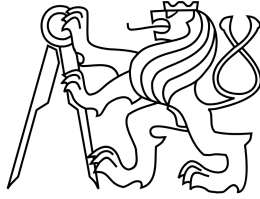


CZECH TECHNICAL UNIVERSITY IN PRAGUE
FACULTY OF ELECTRICAL ENGINEERING
DEPARTMENT OF CONTROL ENGINEERING



MASTER'S THESIS

**Measures and LMIs for Optimal Control of
Piecewise-Affine Dynamical Systems**

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DIPLOMA THESIS ASSIGNMENT

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Guidelines:

The approach consists of formulating optimal control problems as convex linear programming problems on the space of measures, in turn solved via a hierarchy of convex semidefinite programming (SDP) or linear matrix inequality (LMI) problems.

In the case the dynamics are piecewise affine functions defined locally in semialgebraic cells, the occupation measure of the trajectory can be decomposed as a convex combination of local occupation measures, one measure for each cell. The project consists of

- 1) developing an algorithm to carry out optimal controller design
- 2) validating the algorithm on benchmark problems;
- 3) on the theoretical side, investigating the dual formulation, namely the convergence of the dual LMI to the value function, i.e. the viscosity solution of the Hamilton-Jacobi-Bellman partial differential equation.

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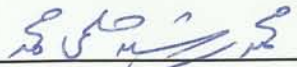
Abstract

The project considers the class of deterministic continuous-time optimal control problems (OCPs) with piecewise-affine (PWA) vector fields and polynomial data. The OCP is relaxed as an infinite-dimensional linear program (LP) over space of occupation measures. The LP is then written as a particular instance of the generalized moment problem which is then approached by an asymptotically converging hierarchy of linear matrix inequality (LMI) relaxations. The relaxed dual of the original LP gives a polynomial approximation of the value function along optimal trajectories. Based on this polynomial approximation, a novel suboptimal policy is developed to construct a state feedback in a sample-and-hold manner. The results show that the suboptimal policy succeeds in providing a stabilizing suboptimal state feedback law that drives the system relatively close to the optimal trajectories and respects the given constraints.

Declaration

I hereby attest that all content presented in this thesis is a result of my own work whereas all used previous material and references are duly indicated.

Prague, May 18, 2012


Mohamed Rasheed-Hilmy Abdalmoaty

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Chapter 1

Introduction

In this chapter, we first introduce, in general setting, the mathematical optimal control theory. The motivation is to fix notations and definitions in section 1.1 while formulating the general deterministic continuous-time optimal control problem. The second section 1.2 states project goals and original contributions. The chapter ends with an outline of the document structure.

1.1 Optimal control theory

The objects under study are dynamical control systems modeled by ordinary differential equations (ODE) that take the form

$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x_0, \quad t \in [0, T] \quad (1.1)$$

in which the map $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is a vector field modeling the controlled dynamics. Vector x denotes the dynamical state and is allowed to assume values in a set $X \subseteq \mathbb{R}^n$, with time derivative $\dot{x} \in \mathbb{R}^n$ governed by f . The independent time variable t is between 0 and terminal time T . Vector u denotes the control and takes its value in a set $U \subseteq \mathbb{R}^m$. Both the state and the control are functions of time, namely $x = x(t)$, $u = u(t)$. At the initial time $t = 0$, the state takes the initial value x_0 in a set $X_0 \subseteq \mathbb{R}^n$. Similarly, at the terminal time, the state takes the terminal value x_T in a set $X_T \subseteq \mathbb{R}^n$. To define what is meant by a solution or a response of the control system, let us fix the control to a constant value, say $u(t) \equiv a \in U$. A solution $x(t)$ of the control system in equation (1.1) (also called state trajectory, or flow) over the interval $[0, T]$ is an absolutely continuous function of time that is differentiable almost everywhere such that

$$x(t) = x_0 + \int_0^t f(x(s), a) ds.$$

It is clear that the state trajectory can be “controlled” by changing the constant value a through time by some function $u(t)$, called control trajectory. The existence and uniqueness of such solutions can be guaranteed under some regularity assumptions imposed on both the vector field and the control function. In cases where the vector field is time independent, assuming that the map $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is locally Lipschitz is sufficient. We assume that $f(x(s), a)$ is a measurable function of s . The state trajectory $x(t)$ is then absolutely continuous in t . We say that the pair of state and control trajectories $(x(t), u(t))$ is admissible, if when starting at x_0 the trajectories stay in $X \times U$ over $[0, T]$. The control functions that generate admissible trajectories are called admissible control functions.

Define for each initial state and each admissible control function the following cost functional (also called performance measure),

$$J(x_0, u(t)) = L_T(x_T) + \int_0^T L(x(t), u(t))dt, \quad \forall (x(t), u(t)) \in X \times U \text{ on } [0, T] \quad (1.2)$$

where the first term $L_T : \mathbb{R}^n \rightarrow \mathbb{R}$ is called the terminal cost, and the integral is called the running cost. The function $L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is called the Lagrangian, and it gives the running cost per unit time. The optimal control problem is the problem of finding, if possible, an admissible control function for the control system (1.1) that minimizes the cost functional (1.2) among all other admissible control functions in a set \mathcal{U} . Is it always true that we can find such optimal control? Can we guarantee the existence and uniqueness? How can we characterize and construct optimal controllers if we know that a solution exists? These are few of many other interesting questions investigated by optimal control theory.

The problem that we have just introduced is composed of two main ingredients, namely a dynamical control system and an associated cost functional. As they appear in (1.1) and (1.2), they form a problem known as Bolza problem. It has two special versions characterized by the cost functional, namely the Mayer problem and the Lagrange problem. In the Mayer problem the Lagrangian is identically zero, $L \equiv 0$, and the total cost is given by the terminal cost. On the other hand, in the Lagrange problem, the terminal cost is identically zero, $L_T \equiv 0$, and the total cost is given by the running cost. Under some regularity assumptions, it is possible to reformulate a Lagrange problem into a Mayer problem and vice versa.

In control engineering applications, the proper definition of the cost functional is usually a difficult task. The cost functional models the required performance specifications. It depends on different factors, like how fast the system is needed to be? How much energy is available for certain maneuver? What is the role of the final time and final state? It is clear that the cost functional depends on the choice of L , L_T , the systems trajectories, but also on the time interval $[0, T]$. The initial state is fixed and given with the control system, while the terminal time T , and terminal state x_T are usually a design parameter depending on the required performance. It is formulated by setting a target set $M \times X_T$, where $M \subseteq [0, T]$. This allows different terminal conditions possibilities. For example, If both M and X_T are singleton we get a fixed-time fixed-endpoint problem. If M is an interval and X_T is a singleton we get a free-time fixed-endpoint problem. A special case is when $M = [0, +\infty)$ and $X_T = X$. Such target set gives a free-time free-endpoint problem. For this case to make sense, the cost functional has to be defined properly in order to rigorously fix when the trajectories are meant to be terminated. In this report we will consider free-time problems in which the terminal time T is a decision variable.

In general, we can classify the mathematical theory of optimal control into two closely related major approaches, namely dynamic programming and variational methods like Pontryagin Maximum principle (PMP). Both methods rely on the same basic idea of reformulating the dynamical optimization problem of optimal control into a set of static optimization problems, one for each time unit, associated with a system of differential equations of two unknown vectors. The first vector is the dynamical state of the control system, and the second is an adjunct vector. This additional variable links together the individual static optimization problems over time. In dynamic programming, the auxiliary variable is the value function, that is a function of the state of the system based on the principle of optimality. In variational approaches, Lagrange multipliers are introduced as the auxiliary variable.

The value function of dynamic programming gives the minimal (optimal) total cost of arriving at the target set among all admissible trajectories starting from a given initial condition. The main observation for the approach is that the value function is the unique solution of a first-order partial differential equation (PDE) of a certain Hamilton-Jacobi type. This

PDE is known as Hamilton-Jacobi-Bellman (HJB) equation. It turns out that solving the HJB equation provides a necessary and sufficient condition for the global optimality of the control function. In addition it helps in synthesizing an optimal feedback control. Unfortunately, in full generality, this PDE is very hard to solve and solutions are not well defined unless we interpret solutions in a generalized sense like viscosity solutions for example. The PMP, on the other hand, can be seen as an extension of the classical calculus of variations. It leads to necessary conditions and results in open-loop solutions.

1.2 Project goals and original contributions

The objective of this research project is to develop a novel algorithm based on the recently developed techniques of occupation measures, see [18], to solve continuous-time optimal control problems where the dynamical control system is piecewise-affine (PWA). A piecewise-affine control system is understood to be a nonlinear control system whose vector field is piecewise-affine in the state with probably multiple equilibrium points. This means that the state space is partitioned into a set of regions or “cells” such that the dynamics in each cell is affine in the state.

The approach consists of formulating the optimal control problem as an infinite-dimensional convex optimization problem (Linear Program - LP) on the space of occupation measures, which is then solved via a converging hierarchy of convex Semidefinite Programming (SDP) problems also called Linear Matrix Inequalities (LMIs). In the special case where the dynamics is PWA defined locally in semialgebraic cells, polytopes for example, the occupation measure of the trajectories can be decomposed as a convex combination of local occupation measures, one for each cell. The dual formulation of the original infinite-dimensional LP problem on occupation measures can be written in terms of Sum-of-Squares (SOS) polynomials. The relaxed dual hierarchy of LMIs yields a converging viscosity subsolutions of the HJB PDE corresponding to the optimal control problem. This gives a good smooth approximating value function along optimal trajectories, which can be used to synthesize an admissible suboptimal feedback control law.

The original contribution of this project is the development of a novel suboptimal feedback synthesis strategy for continuous-time optimal control problems of PWA systems based on the dual variables of the relaxed occupation measure formulation. It adapts the sample-and-hold solution concept. The delivered feedback policy respects both state and input constraints. It is able to provide a near-optimal stabilization of a set of initial states which may require a discontinuous feedback according to the regularity of the control system. It is also able to handle multiple equilibria.

The project consists of the following points:

1. Developing a constructive algorithm to carry out optimal controller design systematically for these PWA dynamics using the polynomial approximation of the value function.
2. Validating the algorithm on a collection of benchmark optimal control problems.
3. Investigating the dual formulation of the measures LP problem, namely the convergence of the dual variables in the LMI hierarchy to the value function viscosity solution of the HJB PDE.

1.3 Document structure

This report is organized in five chapters. Chapter 2 is divided into three sections that attempt to introduce and cover recent developments in the relevant domains. The first section,

2.1, is devoted to continuous-time PWA dynamical systems and the available techniques for controllers design. The second section, 2.2, introduces the dynamic programming approach for optimal control, and the viscosity solutions of PDEs. The last section, 2.3, deals with the concepts of measures and moments theory in addition to their role in solving the Generalized Moments Problem (GMP). Chapter 3 extends the techniques of measures to address optimal control problems of PWA dynamical systems. In section 3.2, we develop the suboptimal strategy assuming the availability of polynomial approximation of the value function. Chapter 4 demonstrates the developed suboptimal strategy on several numerical examples. Finally, chapter 5 concludes the project and outlines open questions and opportunities for further research.

Chapter 2

Background

The chapter seeks to cover background from the relevant domains. It starts by presenting the mathematical model on which further developments are based, then discusses the available controller synthesis techniques for continuous-time PWA systems. The dynamic programming approach is then introduced. We establish that the value function is the unique viscosity solution of the HJB terminal-value problem. The chapter is ended with a section on measures, their moments and their role in solving global nonlinear nonconvex optimization problems. All the results presented here will be needed in the following chapters for the subsequent developments.

2.1 Continuous piecewise-affine dynamical systems

2.1.1 PWA models

Piecewise-affine systems is a large modeling class for nonlinear systems. It can naturally arise from linear systems in the presence of saturation or from simple hybrid systems with state-based switching where the continuous dynamics in each regime is linear or affine [26]. Many engineering systems fall in this category like power electronics converters to mention one example. In addition, common electrical circuits components as diodes and transistors are naturally modeled as piecewise linear elements. On the other hand, PWA systems are used to approximate large class of nonlinear systems as in [15], [22], and [6]. These approximations are then used to pose the controller design problem of the original nonlinear system as a robust control problem of uncertain nonlinear system [24].

In this section we introduce the PWA model which will be used for our further development. The project focuses exclusively on continuous-time PWA systems. The state-space is assumed to be partitioned into a number of regions X_i called cells, such that the dynamics in each cell takes the form

$$\dot{x} = A_i x + a_i + B_i u \quad \text{for } x \in X_i, \quad i \in I \quad (2.1)$$

where the set I is the set of cell indices, and the union of all cells is $X = \cup_{i \in I} X_i \subset \mathbb{R}^n$. The global dynamics of the system depends on both the cells and the corresponding local dynamics. The matrices A_i , a_i , and B_i are time independent. For any arbitrary cell indexed by $i \in I$, the matrix A_i is a square matrix of size $n \times n$ and is called the state matrix. The constant vector a_i is called the bias and it has the same length as the state vector. The matrix B_i is called the input matrix and it is of size $n \times m$, where m is the length of the control vector u . In full generality, the geometry of the regions X_i can be defined arbitrarily. However, for the purpose of this project, the cells are assumed to be compact basic semialgebraic sets, for example polytopes (intersection of finite number of half spaces). The cells have disjoint interiors and are allowed to share boundaries as long as the solution

has measure zero on that boundary. There are many notions of solutions for PWA systems with different regularity assumption on the vector field. The concern here is to ensure the uniqueness of the trajectories. Systems with discontinuous right-hand side can have attracting sliding modes, non-unique trajectories, or trajectories may not even exist in the classical sense [8]. This can happen for example if some important features of the modeled physical system were ignored while developing the model. More on PWA models and solution concepts can be found in [14].

For the work done in this thesis, we assume that the PWA system is well-posed in the sense that it generates a unique trajectory for any given initial state. This is guaranteed if we assume PWA continuous functions. This is usually the case if the model is the result of approximating a nonlinear function.

2.1.2 Optimal control of continuous-time PWA systems

Piecewise-affine systems are a special class of nonlinear systems. However most of the nonlinear control theory does not apply to PWA systems because it requires certain smoothness assumptions. On the other hand, linear control theory cannot be simply employed due to the special properties of PWA systems inherited from nonlinear systems.

An optimal control problem of continuous-time PWA systems is an optimal control problem, as introduced in section 1.1, in which the vector field f is piecewise-affine. Usually, OCPs of PWA systems are Lagrange problems where the state space is partitioned into a finite number of cells. In addition to the dynamics, the cost functional can be also defined locally in each cell. As any other OCP, the objective is to find optimal trajectories starting from an initial set and terminating at a target set that minimize the running cost and respect some input and state constraints.

One main motivation behind this research is the absence of optimal control synthesis methods for continuous-time piecewise-affine systems with both input and state constraints. The motivation goes behind optimality when there is a need to find suitable tools for the design of stabilizing feedback controllers under some constraints. Over the last few years, there were several attempts addressing the synthesis problem for continuous-time PWA systems. The techniques are based on analysis methods and use convex optimization programs. These methods assume a quadratic cost function and result in a state-based switched linear controllers. For example, in [12] a piecewise linear state-feedback controller synthesis is done for piecewise-linear systems by solving convex optimization problem involving LMIs. The method is based on constructing a globally quadratic Lyapunov functions such that the closed loop system is stable. Similarly, in [21] a quadratic performance index is suggested to obtain lower and upper bounds for the optimal cost using any stabilizing controller. However, the optimal controller is not computed. In [23], the work done in [12] is extended to obtain dynamic output feedback stabilizing controllers for piecewise-affine systems. It formulates the search for piecewise-quadratic Lyapunov function and piecewise-affine controller as a nonconvex Bilinear Matrix Inequality, solved only locally by convex optimization methods.

To the best of the author's knowledge there are no available methods for synthesis of optimal controllers in continuous-time for PWA systems that do not restrict the controller to be piecewise linear or do not require the performance index to be quadratic. The technique presented in this research project provides a systematic approach to synthesize suboptimal state feedback control law for continuous-time PWA systems with multiple equilibria. The suboptimal controller respects any state or input constraints, while minimizing a piecewise cost functional that is not necessarily quadratic.

2.2 Dynamic programming and HJB PDE

In this section we study the value function as a function of state for a general free-time optimal control problem. In the introductory chapter 1, the value function was thought of as the optimal cost to reach some target set given that we start at some initial conditions. The section begins by stating the principle of optimality. Then we proceed, assuming the differentiability of the value function, by rewriting the optimality condition as an HJB PDE relating the optimal values for different states. We show that solving the PDE for the value function gives both necessary and sufficient conditions for optimality. Next we deal with the difficulties that would arise if the value function were non-smooth by adapting a generalized notion of solution namely, viscosity solutions. The material in this section is covered in [4], [25], [1], [19], and [11].

2.2.1 Optimality conditions

The dynamic programming approach is based on a very simple and intuitive idea called the principle of optimality. The name is due to Richard Ernest Bellman (August 26, 1920 - March 19, 1984), an American applied mathematician. The principle states that the optimal policy has the property that no matter what the initial state and initial control are, the control (decision) of the remaining period must constitute an optimal policy for the current state. This means that along optimal trajectories, no crosscuts are allowed. This principle suggests that the optimal policy can be constructed in a stepwise manner starting at the final state and going backward. To formulate the principle in mathematical terms consider a free-time free-endpoint Bolza problem with a cost

$$J(x_0, u(t)) := L_T(x_T) + \int_0^T L(x(s), u(s)) ds$$

where the arguments of $J(\cdot)$ denote that the cost depends on the initial conditions x_0 as well as the control trajectory over the time interval $[0, T(u)]$. The trajectories obey the general dynamical control system defined by the ODE $\dot{x}(t) = f(x(t), u(t))$ with $x(0) = x_0$. Furthermore, consider any intermediate time instant τ such that $0 \leq \tau < T$. The cost starting at $x(\tau)$ is then given by

$$J(x(\tau), u(t)) = L_T(x_T) + \int_{\tau}^T L(x(s), u(s)) ds$$

in which $x(\cdot)$ is the solution of

$$\dot{x}(t) = f(x(t), u(t)) \quad \tau \leq t < T$$

and u any admissible control over the time interval $[\tau, T]$. Define the value function

$$v^*(x) := \inf_{u, T} J(x, u) \quad \forall x \in X \quad (2.2)$$

such that it satisfies the terminal conditions

$$v^*(x_T) = L_T(x_T). \quad (2.3)$$

The optimality conditions are then given by the relation

$$v^*(x(\tau)) = \inf_u \left\{ \int_{\tau}^{\tau+h} L(x(s), u(s)) ds + v^*(x(\tau+h)) \right\} \quad (2.4)$$

for every $h > 0$ such that $\tau + h \leq T$, and $x \in X$. The trajectory $x(\cdot)$ corresponds to the admissible control u on time range $[\tau, \tau + h]$ with an initial state $x(\tau)$. Using the infimum in the definition of the value function, means that the existence of optimal control is not assumed. For a justification of the relation, refer to [11] or [4].

2.2.2 Hamilton-Jacobi-Bellman equation of optimal control

The value functions is an auxiliary object connecting a family of optimization problems. This can be seen from the principle of optimality relation in (2.4). The value function appears on both sides giving the optimal cost under optimal control trajectory for different time-state values (events) separated by some time h . This dynamical relation can be rewritten as a first-order partial differential equation. So far we did not impose any regularity assumption on the value function. For the current purpose, we assume that $v^*(\cdot) \in \mathcal{C}^1$, i.e. v^* is continuously differentiable, an assumption that will be relaxed later. Under this assumption we can use the first order Taylor expansion to rewrite (2.4) as $h \rightarrow 0$ and arrive at the following stationary PDE for the value function

$$\inf_{u \in U} \{ \nabla v^*(x) \cdot f(x, u) + L(x, u) \} = 0 \quad \forall x \in X \quad (2.5)$$

where ∇ denotes the gradient. This is the Hamilton-Jacobi-Bellman equation of optimal control. We note the appearance of the gradient of the value function. The HJB equation is satisfied over the time interval $[0, T]$ for every admissible state x by the value function. The terminal conditions are reflected by (2.3). The HJB equation can be written in compact form using the notion of control Hamiltonian:

$$H(x, u, p) := p \cdot f(x, u) + L(x, u) \quad (2.6)$$

which is seen as a function of $(x, u) \in X \times U$, and a variable $p \in \mathbb{R}^n$ called the co-state or adjunct vector. The co-state vector is to be interpreted as the gradient of the value function with respect to the state. It measures the sensitivity of the optimal cost to changes in x . The HJB equation in terms of H reads:

$$\inf_{u \in U} H(x, u, \nabla v^*) = 0. \quad (2.7)$$

To show that the HJB equation gives a necessary condition we assume that the value function is achieved by some control trajectories u^* and x^* starting from the initial state x_0 . Applying the optimal control starting from this initial state minimizes the control Hamiltonian along the trajectory, and therefore yields

$$H(x^*, u^*, \nabla v^*) = \inf_{u \in U} H(x^*, u, \nabla v^*) \quad (2.8)$$

and we conclude that the condition provided by the equation is necessary for optimality. This condition is also sufficient for optimality. Under the assumption that the value function is continuously differentiable, we say that there is a smooth solution of the optimal control problem if there is a continuous control $u^*(t)$ for every $x \in X$ such that the minimum of the Hamiltonian is achieved (we also say that the value function is achieved). In this case we can use minimum instead of infimum in the right-hand side of the HJB PDE. To establish this, define $x^*(t)$ over the interval $[0, T]$ to be the state trajectory generated by applying the control $u^*(t)$ starting at the initial state x_0 such that the control attains the minimum of the Hamiltonian. Let $u(t)$ be any admissible control trajectory with a corresponding state trajectory denoted $x(t)$. Assume that the \mathcal{C}^1 function $v(x)$ satisfies the HJB equation with the associated terminal condition. Then for every time $t \in [0, T]$ we have

$$0 \leq \nabla v(x) \cdot f(x, u) + L(x, u)$$

and observing that

$$\dot{v}(x) = \nabla v(x) \cdot f(x, u)$$

brings us to the inequality

$$0 \leq \dot{v}(x) + L(x, u).$$

Integrating the inequality over the time interval $[0, T]$, and using the fundamental theorem of calculus we conclude that

$$v(x_0) \leq L_T(x_T) + \int_0^T L(x(t), u(t)) dt.$$

If we used the trajectories $u^*(t)$, and $x^*(t)$, then by their definitions, the last inequality becomes equality. This shows that no other control trajectories except $u^*(t)$ attains the minimum. Consequently $v(x_0) = v^*(x_0)$, and the control $u^*(t)$ is optimal.

Therefore applying the dynamic programming approach yields the following terminal-value problem for the HJB equation

$$\begin{aligned} \inf_{u \in U} \{ \nabla v^*(x) \cdot f(x, u) + L(x, u) \} &= 0 & \forall x \in X \\ v^*(x_T) &= L_T(x_T) \end{aligned} \quad (2.9)$$

whose solution gives both necessary and sufficient conditions. The bottom line is that if we are able to solve this terminal-value problem, we will arrive at a solution to the optimal control problem. In the next chapter, this problem will be defined for the specific case of having PWA dynamics. In such special case, the problem inherently contains state constraints. We will show that the occupation measures approach is very convenient for solving this problem taking into account both state and input constraints.

2.2.3 Viscosity solution of HJB PDE

It is generally not true that the value function of an optimal control problem is a continuously differentiable function. Actually it is quite common for the simplest problems with bounded controls to have non-differentiable value functions. For example consider the following free-time fixed-endpoint OCP

$$\begin{aligned} \min_{u, T} \quad & \int_0^T dt \\ \text{s.t.} \quad & \dot{x} = u \\ & x(0) = x_0, \quad x(T)^2 = 1 \\ & x \in [-1, 1], \quad u \in [-1, 1]. \end{aligned}$$

The initial state $x(0)$ assumes any admissible value, and the terminal state $x(T)$ is either -1 or 1 . The control Hamiltonian is

$$H(x, u, \nabla v^*) = \nabla v^* \cdot u + 1,$$

where v^* is the value function of the OCP. The HJB PDE is written as follows

$$\begin{aligned} \min_{u \in [-1, 1]} \{ \nabla v^*(x) \cdot u + 1 \} &= 0 & \forall x \in [-1, 1] \\ v^*(x(T)) &= 0. \end{aligned}$$

Since u is constrained, the minimum of the control Hamiltonian is attained on the boundary of the control interval $[-1, 1]$ and the optimal control is given by

$$u^* = -\text{sign}(\nabla v^*)$$

Using this characterization for the optimal control in the HJB PDE, we get the following Eikonal PDE

$$-(\nabla v^*)^2 + 1 = 0$$

with boundary conditions $v^*(-1) = v^*(1) = 0$. The solution of this PDE gives the value function $v^*(x) = -|x| + 1$. Therefore the slope of the value function is $+1$ whenever $x \in [-1, 0)$ and is -1 whenever $x \in (0, 1]$. The slope is changed only once at $x = 0$ to meet the boundary conditions. At that point where the slope changes, the derivative of the value function is not defined. In such case the value function fails to be differentiable and the HJB PDE does not admit any classical solution (\mathcal{C}^1 solution). Instead, we can talk about weak solutions. Clearly, there are infinitely many weak solutions to the PDE which are merely continuous functions satisfying the boundary conditions with slope either $+1$ or -1 in the interval $[-1, 1]$. However, there is only one unique solution that corresponds to the value function v^* of the OCP. How can one determine such solution? The recent developments in non-smooth analysis and the notion of viscosity solution provide an answer. The notion of viscosity solutions is due to [10], and the properties of the solution is due to [9]. The connection to the HJB equation of optimal control can be found in [4]. Our motivation is to define a unique weak solution of the terminal value problem in (2.9). The important results that we want to state can be summarized in three points:

- A bounded uniformly continuous function u is a unique weak solution, called viscosity solution, of the terminal-value problem provided that it satisfy the boundary conditions and two inequality conditions (see definition of viscosity solution below).
- The value function of the OCP is the unique viscosity solution.
- The definition of viscosity solution provides a convergence limit to some numerical optimization problem defined in the next chapter. If we can guarantee that the numerical method converges to a viscosity solution, it means that we converge to the value function of the optimal control problem.

The terminal-value Cauchy problem in (2.9) can be written in a compact form using the definition of the control Hamiltonian (2.6):

$$\begin{aligned} \min_{u \in U} H(x, u, \nabla v) &= 0 & \forall x \in X \\ v(x_T) &= L_T(x_T). \end{aligned} \tag{2.10}$$

A viscosity solution of (2.10) is defined as a bounded uniformly continuous function v such that:

- (i) It satisfies the boundary condition $v(x_T) = L_T(x_T)$
- (ii) For each test function $w \in \mathcal{C}^\infty(X)$

$$\begin{aligned} &\text{If the function } v - w \text{ has a local maximum at a point } x \in X \\ &\text{then} \\ &\min_{u \in U} H(x, u, \nabla w) \geq 0 \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} &\text{If the function } v - w \text{ has a local minimum at a point } x \in X \\ &\text{then} \\ &\min_{u \in U} H(x, u, \nabla w) \leq 0 \end{aligned} \tag{2.12}$$

We say that v is a viscosity subsolution, if inequality (2.11) is satisfied. Similarly, we call v a viscosity supersolution if inequality (2.12) is satisfied. The function is said to be a viscosity solution if it both a viscosity subsolution and a viscosity supersolution. This is the key behind the uniqueness of the viscosity solution.

It can be shown (see [4, Ch8] or [11, Ch10]) assuming that the OCP functions f , L , and L_T are bounded and Lipschitz continuous, that the OCP value function v^* is:

- (i) a bounded and Lipschitz continuous function,
- (ii) the unique viscosity solution of the terminal-value problem (2.10) according to the definition presented above. Therefore, the value function satisfies the two inequalities (2.11) and (2.12).

Knowing this, we mention without going into details that the HJB theory which is developed in subsection 2.2.2 can be generalized rigorously under the framework of viscosity solutions.

The main conclusion of this section is that the dynamic programming approach to optimal control problems gives both necessary and sufficient conditions. The value function is the unique viscosity solution of the HJB terminal-value problem. In the next chapter, we will show how to arrive at approximate solutions of this problem, and how it can be helpful for optimal control design.

2.3 Measures, moments and LMIs

In this section, we introduce in simple terms the notions of measures and moments as well as some recent results from real algebraic geometry that are needed for the subsequent developments. The level of abstraction is kept at a point relevant to that purpose. First we define measures and their moments. Then we describe in short the Generalized Moment Problem (GMP) with polynomial data and its dual problem. We follow by reporting the role of moments in solving the GMP and the connection to Linear Matrix Inequalities (LMIs). The section ends with the application of these results to solve global static polynomial optimization. The technique will be used in the following chapter as a part of the suggested suboptimal policy for PWA optimal control problems.

2.3.1 Measures

A measure μ is a function assigning a real number to a set. The support of a measure is the smallest closed set on which the measure has a nonzero value. Let \mathbb{R}^* be the extended real line and let $K \subset \mathbb{R}^n$ be the support of the measure $\mu : K \rightarrow \mathbb{R}^*$ so that $\mu(\mathbb{R}^n/K) = 0$. The action of the measure μ on the support K is denoted:

$$\mu(K) = \int_K d\mu(x)$$

where $x \in \mathbb{R}^n$. We will be considering nonnegative measures only, namely measures that returns either zero or a positive number, and we note $\mu \geq 0$ meaning that $\mu(K) \geq 0$ for all sets $K \subset \mathbb{R}^n$. The number returned by the measure can be seen as an indication of the size of the input set. This gives a generalization of the notion of length, area, and volume.

Perhaps the most common measure is the Lebesgue measure typically denoted $\mu(dx) = dx$ which is used to construct the Lebesgue integral. On \mathbb{R}^n the Lebesgue measure returns the conventional length, area, and volume of Euclidean geometry to bounded subsets of the Euclidean space of dimension 1, 2, and 3 respectively. It returns $+\infty$ on unbounded sets. Another example of a measure from probability theory that may act on unbounded sets is the Hermite measure, $\mu(dx) = e^{-x^T x} dx$, which is defined with an exponentially decaying

function. Any nonnegative measure such that $\mu(K) = 1$ is called a probability measure. A Dirac measure $\mu(dx) = \delta_{x^*}$ is a probability measure defined on a single isolated point x^* such that $\mu(\{x^*\}) = 1$. An atomic measure is a nonnegative linear combination of several Dirac measures defined on a finite number of isolated points.

Measures can also be indifferently seen as a very specific class of distributions acting on the set of infinitely differentiable functions with compact domains (or that vanish at infinity). One of the Riesz representation theorems identifies measures with continuous linear functionals. Given a function f of bounded variation on a compact set K , we can define the so-called Stieljtes measure as its derivative, with the action

$$v \mapsto \int_K v df$$

on smooth continuous functions v , which are also called test functions. This provides a connection between measures and functions of bounded variations and allows for measures classification. The classification is done by a counterpart of Lebesgue decomposition of functions of bounded variations. Any measure μ can be decomposed and written as the sum of three measures of different types. Every measure μ is expressible as

$$\mu = \mu_{AC} + \mu_{SC} + \mu_{SD}$$

where μ_{AC} is an absolutely continuous measure, μ_{SC} is a singularly continuous measure, and μ_{SD} is an atomic measure. This explains geometrically what can be modeled using measures. In other words, using measures we can deal with functions that are absolutely continuous, singular (Cantor-like), or jump functions. We can deal with sets which are smooth, irregular (fractal), or sets of isolated points (discrete). A question that directly follows is how we can deal with measures themselves which are quite abstract objects. Measures can be manipulated using their moments.

The moments of a measure is a sequence of real numbers that can be defined by setting test functions to monomials. A monomial of n variables is a multivariate function defined with a multi-index notation as $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$ where $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$, where \mathbb{N}^n refers to the set of nonnegative integers vectors of size n . The degree of the monomial is $|\alpha| = \sum_{i=1}^n \alpha_i$. The α -th moment of a measure μ is the real value

$$y_\alpha = \int_K x^\alpha d\mu(x).$$

The classical moment problem is defined as follows: given a closed set $K \subset \mathbb{R}^n$, and an infinite sequence $y = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, can we find a representing nonnegative measure μ supported on K , such that

$$y_\alpha = \int_K x^\alpha d\mu(x), \quad \forall \alpha \in \mathbb{N}^n?$$

What are the conditions on the sequence y which if satisfied guarantee the existence of a representing nonnegative measure supported on K ? In the case of nonnegative measures, it is known that the conditions are related to a characterization of positive polynomials on the given support. Fortunately, in the special case where the support is a compact basic semi-algebraic set, Putinar's Positivstellensatz, see [17], provides such a characterization. Before stating the conditions, we should define the notion of moment matrix and localizing matrix. We take a polynomial p in n variables as a finite linear combination of monomials. Let $\mathbb{R}[x]_d$ be the real vector space of real-valued multivariate polynomials with d as the maximum possible degree. The standard basis element of $\mathbb{R}[x]_d$ is the monomial x^α of degree less than

or equal to d . Then we write the polynomial $p \in \mathbb{R}[x]_d$ as

$$p(x) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha$$

where $p_\alpha \in \mathbb{R}$ are real coefficients.

Given any infinite sequence $y = (y_\alpha)$ of real numbers with $\alpha \in \mathbb{N}^n$, define the linear functional $\ell : \mathbb{R}[x] \rightarrow \mathbb{R}$ that maps polynomials to real numbers as follows

$$p(x) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha \quad \mapsto \quad \ell_y(p) = \sum_{\alpha \in \mathbb{N}^n} p_\alpha y_\alpha.$$

The moment matrix $M_d(y)$ is defined to be a square matrix of dimension $\binom{d+n}{n}$ filled with the first $2d$ moments. Its entries are indexed by the multi-indices β (for rows) and γ (for columns) such that the entry of the β row and γ column is the $(\beta + \gamma)$ -th moment:

$$[M_d(y)]_{\beta, \gamma} = \ell_y(x^{\beta+\gamma}) = y_{\beta+\gamma}, \quad \forall |\beta| + |\gamma| \leq 2d, \quad \beta, \gamma \in \mathbb{N}^n.$$

A necessary condition for the given sequence to have a representing measure μ regardless of the support is given by the restriction

$$M_d(y) \succeq 0, \quad \forall d$$

which means that matrix $M_d(y)$ is positive semidefinite, i.e. its eigenvalues are all real nonnegative. By definition it is clear that the moment matrix $M_d(y)$ depends linearly on y . The necessary condition therefore requires that all the eigenvalues of $M_d(y)$ be nonnegative. This is seen as a constraint on the sequence y . To give a simple example of the construction of the moment matrix we set the number of variables $n = 1$ and the order $d = 3$. We then write the moment matrix filled with moments up to y_6 :

$$M_3 = \begin{bmatrix} 1 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \\ y_2 & y_3 & y_4 & y_5 \\ y_3 & y_4 & y_5 & y_6 \end{bmatrix}.$$

We note that the anti-diagonal elements are constant. This corresponds to the Hankel structure. The construction is easily generalized to several variables. Assume $n = 2$, the 2nd order moment matrix is then a quasi-Hankel square matrix with six rows and six columns filled with moments up to the fourth moment using the indices $(00, 10, 01, 20, 11, 02, 30, 21, \dots)$. The necessary condition can be also sufficient if the given sequence is such that $|y_\alpha| \leq 1$, $\forall \alpha$. In this case the given sequence has a representing measure μ with support contained in the box $[-1, 1]^n$. This means that if the moment matrix is positive semidefinite and we placed some bounds on the given moment sequence, we get information about the support of the representing measure.

What remains now is the conditions on the given sequence to have a representing measure supported on a specific given set K (the so-called K -moment problem). Assume finite number of given polynomials $p_k \in \mathbb{R}[x]$, with the index $k = 1, \dots, m$ and define $K \subset \mathbb{R}^n$ as a given compact basic semi-algebraic set such that

$$K := \{x \in \mathbb{R}^n \mid p_k(x) \geq 0, k = 1, \dots, m\}. \quad (2.13)$$

The localizing matrix is defined with respect to the given sequence y and a polynomial $p_k(x) = \sum_{\alpha} p_{k\alpha} x^\alpha$ in a similar fashion as we did with the moment matrix. Define the localizing matrix to be the matrix $M_d(p_k y)$ of order d , dimension $\binom{d+n}{n}$, and entries

$$[M_d(p_k y)]_{\beta, \gamma} = \ell_y(p_k(x) x^{\beta+\gamma}) = \sum_{\alpha} p_{k\alpha} y_{\alpha+\beta+\gamma}, \quad \forall |\alpha| + |\beta| + |\gamma| \leq 2d.$$

Accordingly, the localizing matrix can be seen as a linear combination of the moment matrix and shifted moment matrices. In 1993 in [20], it was shown that if $M_d(y) \succeq 0$, and $M_d(p_k y) \succeq 0$, $\forall k, \forall d$, the given sequence corresponds to a representing measure μ with a support contained in the compact basic semi-algebraic set K . These conditions are both necessary and sufficient. They are usually called Putinar's conditions.

2.3.2 Linear matrix inequalities

The connection with LMIs is given directly by Putinar's conditions given in subsection 2.3.1. Geometrically, the conditions constrain the moment matrix and the localizing matrix to the convex set of positive semidefinite matrices, which is a convex cone. Remember that both the moment matrix and the localizing matrix are linear symmetric functions of the vector y . Thus, if the condition is satisfied, it means that the intersection of the subspace defined by these matrices and the semidefinite cone is not empty. Consequently, we can find a real vector y that belongs to that linear section which has a representing measure on the given support. This condition is what we call Linear Matrix Inequality (LMI), see [2]. So the bottom line here is that Putinar's conditions are LMI constraints on the given sequence y . This allows the manipulation of measures through their moments with the help of LMI and semidefinite programming (SDP) which is linear programming on the cone of positive semidefinite matrices.

2.3.3 Generalized moment problem

In the next subsection, the global static polynomial optimization problem will be written using measures as a special instance of an infinite-dimensional linear optimization problem. Later in chapter 3, the OCP will be formulated in terms of occupation measures and then written, similarly, as a particular instance of an infinite-dimensional linear optimization problem. These infinite-dimensional linear programs are called Generalized Moment Problems (GMPs). In the following we describe the GMP and its dual problem.

For any given support K , let $\mathcal{M}(K)$ be the space of signed Borel measures μ supported on K . Let the scalar functions $p_k : K \rightarrow \mathbb{R}$ be measurable for $k = 0, 1, \dots, m$.

For given real sequences y_k , $k = 1, \dots, m$, the GMP is defined as an infinite-dimensional LP as follows:

$$\begin{aligned} p^* &= \inf_{\mu \in \mathcal{M}(K)} \int_K p_0(x) d\mu(x) \\ \text{s.t.} \quad &\int_K p_k(x) d\mu(x) = y_k, \quad k = 1, \dots, m \\ &\mu \geq 0. \end{aligned} \tag{2.14}$$

It can also be defined with inequality constraints, or with both equality and inequality constraints. The GMP has powerful modeling capabilities. Many problems from diverse application areas of mathematics, economics and engineering can be written as instances of the GMP. Note that if the function p_0 is identically zero, $m = 1$ and p_1 is a polynomial, optimization problem (2.14) becomes the classical moment problem, which consists in finding a finite positive representing measure supported on K for a given real sequence y_1 . In the special case where the support K is a compact basic semi-algebraic set and the scalar functions are polynomials, Putinar's conditions give necessary and sufficient conditions on the optimal solution as explained in the previous subsection.

The dual problem of the infinite-dimensional LP (2.14) can be written using Lagrange duality from convex analysis as follows

$$\begin{aligned} d^* &= \sup_{\lambda} \sum_{k=1}^m y_k \lambda_k \\ \text{s.t.} \quad &p_0(x) - \sum_{k=1}^m \lambda_k p_k(x) \geq 0, \quad \forall x \in K \\ &\lambda_k \geq 0, \quad k = 1, \dots, m, \end{aligned} \tag{2.15}$$

where $\lambda \in \mathbb{R}^m$ is the dual variable. In the case where K is a compact basic semi-algebraic set, and the functions p_k are polynomials, the first constraint in (2.15) asks for a positive polynomial (nonnegative) over the support K . Therefore, the theory of moments and the theory of representation of positive polynomials are dual [16], and this duality is captured by convex optimization duality. The problem of ensuring that a polynomial is positive can be traced back to the beginning of the 20th century and is known to be a hard condition. In the next subsection, we show how to relax this condition by decomposing the polynomial as sum-of-squares (SOS). This reduces the condition to an LMI feasibility problem.

The optimal value d^* of the dual problem is by definition the best lower bound on the primal optimal value p^* , and the inequality $d^* \leq p^*$ always holds. This property is called the weak duality. If the inequality is strict, we say that a duality gap exists. If there is no duality gaps, the equality $d^* = p^*$ holds, and we call this property the strong duality. According to Slater's condition [3], strong duality holds for problems (2.14) and (2.15) if the problem is strictly feasible. The problem is strictly feasible if we can find a decision variable that belongs to the interior of the constraint cone. However, there are some special cases where the Slater's condition is not necessary for strong duality of the problems (2.14), and (2.15) to hold. This happens if the support K is compact, the integrand of the objective function p_0 is bounded and upper semi-continuous on the support K , and $\forall k = 1, \dots, m$ the functions p_k defining the support are continuous. Under these assumptions, strong duality holds, and if the problem is feasible, the minimum is attained. Furthermore the optimal measure is an atomic measure. For a detailed discussion and a proof of these results refer to [17].

2.3.4 Global optimization over polynomials

Consider the problem of globally minimizing a multivariate real-valued polynomial of real coefficients over intersection of finite polynomials equalities and inequalities. This is generally a hard multi-extremal nonconvex polynomial programming problem. We now apply the results discussed in the previous subsections to solve this problem.

Assume a multivariate polynomial $p(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ and let K be a compact subset of the Euclidean space \mathbb{R}^n as defined in (2.13). The constrained polynomial optimization problem

$$\begin{aligned} p_1^* &= \inf_x p(x) \\ \text{s.t. } & x \in K \end{aligned} \quad (2.16)$$

can be written as follows

$$\begin{aligned} p_2^* &= \inf_{\mu \in \mathcal{M}(K)} \int_K p(x) d\mu(x) \\ \text{s.t. } & \mu(K) = 1 \\ & \mu \geq 0 \end{aligned} \quad (2.17)$$

which is a particular instance of the GMP (2.14). In other words we can formulate the general nonlinear optimization problem (2.16) as a linear, hence convex, optimization problem which is however infinite-dimensional. The constraint of the GMP formulation means that we optimize over probability measures. We know from previous subsections, that with the definition of the support K in (2.13), problem (2.17) attains its minimum and has an optimal solution of atomic type. Therefore, if the vector x^* is optimal for problem (2.16), the measure $\mu = \delta_{x^*}$ is an optimal measure for the GMP formulation.

To justify that indeed the two problems are equivalent and that $p_1^* = p_2^*$, note that by the definition of the primal problem $p(x) \geq p_1^*, \forall x \in K$. It is equivalent to write $\int_K p(x) d\mu \geq p_1^*$ with a probability measure. Therefore $p_2^* \geq p_1^* \forall x \in K$. To establish the equality, the reverse inequality has to be true. It can be seen directly from the definition of the dual formulation, where the optimal value $p_2^* \leq \int_K p(x) d\delta_x$, for every and each feasible x . Therefore $p_2^* \leq p_1^*, \forall x \in K$.

The objective function in (2.17) can be written as a linear function of moments of μ by writing the polynomial in terms of a finite linear combination of monomials:

$$p(x) = \sum_{\alpha} p_{\alpha} x^{\alpha}, \quad \forall x \in K.$$

It then gives the following primal formulation

$$\begin{aligned} p^* &= \inf_{\mu} \sum_{\alpha} p_{\alpha} \int_K x^{\alpha} d\mu(x) \\ &= \inf_y \sum_{\alpha} p_{\alpha} y_{\alpha} \\ &= \inf_y \ell(p). \end{aligned} \tag{2.18}$$

Solving the optimization problem over x is the same as solving this primal formulation over an infinite sequence of moments corresponding to a probability measure supported on the constraint of the original problem. Note that the linear problem on moments is infinite-dimensional. In the special case where K is a compact basic semi-algebraic set as in (2.13), Putinar's conditions can be used to generate a hierarchy of LMI relaxations.

For a fixed relaxation order d , the problem is reduced to the LMI

$$\begin{aligned} p_d^* &= \inf_y \ell_y(p) \\ \text{s.t. } & M_d(y) \succeq 0, \\ & M_d(p_k y) \succeq 0, \quad k = 1, \dots, m. \end{aligned} \tag{2.19}$$

With increasing relaxation order, solving LMI (2.19) gives a nondecreasing asymptotically converging sequence of lower bounds on p^* . For larger d , we get more moments in the sequence y (which is finite now), and we optimize over a larger linear section of the semidefinite cone (tighter constraint). The lowest possible relaxation order, call it d_0 , must at least enumerate all the moments appearing in the objective function. In the next relaxation order, $d_1 = d_0 + 1$, the moment and localizing matrices will include additional moments that do not appear in the objective function. This explains why we have convergence such that

$$p_{d_0}^* \leq p_{d_1}^* \leq \dots \leq p_{d_{\infty}}^* = p^*$$

One way to write the dual formulation of the problem (2.16) is by maximizing the lower bound of the polynomial epigraph over K as follows

$$\begin{aligned} d^* &= \sup_{\lambda} \lambda \\ \text{s.t. } & p(x) - \lambda \geq 0, \quad \forall x \in K. \end{aligned} \tag{2.20}$$

The constraint requires polynomial $p(x) - \lambda$ to be positive whenever the polynomials defining the support K , are positive, see (2.13). It is well-known that in one dimension any nonnegative polynomial $p \in \mathbb{R}[x]$ has a sum-of-squares (SOS) decomposition, i.e. it can be written as a finite linear combination of squared (hence positive) polynomials. This is not the case in higher dimensions. In multiple dimension, not every nonnegative polynomial has an SOS decomposition. However, any polynomial that is SOS over some domain is nonnegative over that domain.

The constraint $p(x) - \lambda \geq 0, \forall x \in K$ can be relaxed by using SOS decomposition with unknown multipliers as functions of x . The dual problem can then be written in terms of these unknown multipliers as follows [16]

$$\begin{aligned} d^* &= \sup_q \lambda \\ \text{s.t. } & p(x) - \lambda = \left(\sum_j q_{j0}^2(x) \right) + \sum_k \left(\sum_j q_{jk}^2(x) \right) p_k(x). \end{aligned} \tag{2.21}$$

By fixing the degree of the SOS polynomials to d , the problem can be formulated as a dual hierarchy of LMI problems. To summarize, the global optimization of polynomials is nothing more than LMI problem on moments from the primal side. On the dual side we have an LMI problem on the coefficients of SOS polynomials that represent a positive polynomial on K . For further details and numerical examples, see [16].

Chapter 3

Application of measures to OCPs of PWA systems

3.1 Occupation measures for optimal control of PWA systems

Consider the following general piecewise-affine optimal control problem

$$\begin{aligned}
 v^*(x_0) = & \inf_{T, u} L_T(x_T) + \int_0^T L_i(x(t), u(t)) dt \\
 \text{s.t. } & \dot{x} = A_i x(t) + a_i + B_i u(t), \quad x \in X_i, \quad i = 1, \dots, r \\
 & x(0) = x_0 \in X_0 \subset \mathbb{R}^n, \\
 & x(T) = x_T \in X_T \subset \mathbb{R}^n, \\
 & (x(t), u(t)) \in X \times U \subset \mathbb{R}^n \times \mathbb{R}^m, \quad t \in [0, T].
 \end{aligned} \tag{3.1}$$

Our goal in this chapter and to the end of this report is to use the background presented in the previous chapters to solve the above problem by constructing a state-feedback controller that generates an admissible suboptimal control trajectory. Both the Lagrangian and the terminal cost are assumed to be polynomial maps, namely $L_i \in \mathbb{R}[x, u] \forall i$, and $L_T \in \mathbb{R}[x]$. The PWA system is well-posed and the vector field is locally Lipschitz. The cells X_i and the control set U are assumed to be compact basic semi-algebraic sets

$$X_i \times U = \{(x, u) \in \mathbb{R}^n \times \mathbb{R}^m \mid p_{i,k}(x, u) \geq 0, \quad \forall k = 1, \dots, m_i, \quad i = 1, \dots, r\}.$$

Furthermore we assume that the sets X_0, X_T are all compact basic semi-algebraic sets

$$\begin{aligned}
 X_0 &= \{x \in \mathbb{R}^n \mid p_{0,k}(x) \geq 0, \quad \forall k = 1, \dots, m_0\}, \\
 X_T &= \{x \in \mathbb{R}^n \mid p_{T,k}(x) \geq 0, \quad \forall k = 1, \dots, m_T\},
 \end{aligned} \tag{3.2}$$

3.1.1 Occupation measures and their moments

Occupation measures are measures used to deal with dynamic objects where time is involved. We focus on the application of occupation measures to dynamical control systems where the dynamical objects are ordinary differential equations. For the purpose of this project, occupation measures are used to solve PWA optimal control problems defined in (3.1). The idea is simple and quite similar to what was shown in subsection 2.3.4 for static optimization over polynomials. Starting by the given ODE, we generate a sequence of moments by writing the ODE in terms of occupation measures. We then manipulate the measures through their moments to optimize over system trajectories, using the representation conditions.

Although occupation measures can be defined for any general dynamical systems, we first consider, for simplicity, the uncontrolled autonomous dynamical system defined by the Lipschitz vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and the nonlinear differential equation

$$\dot{x} = f(x), \quad x(0) = x_0. \quad (3.3)$$

The indicator function of the compact subset $X \subset \mathbb{R}^n$ can be defined for the unique solution of the above system as a scalar mapping $\mathbb{I}_X : X \rightarrow \{0, 1\}$ such that:

$$\mathbb{I}_X(x(t)) := \begin{cases} 1, & \text{if } x(t) \in X \\ 0, & \text{if } x(t) \notin X. \end{cases}$$

The occupation measure of the solution over X is simply defined as the time integration of the indicator function over some interval $[0, T]$

$$\mu(X) := \int_0^T \mathbb{I}_X(x(t)) dt.$$

It is important to note that the mass of the occupation measure is the time spent by the solution in the subset X . This means that occupation measure conveys information about the solution of the system. In other words, by its definition through the indicator function, the occupation measure encodes the trajectories of the differential equation in the sense that it can indicate when the solution is within a given subset X . Taking the whole Euclidean space as the support, the mass of the occupation measure is given by the terminal time T . In general we should make sure that T is finite, otherwise the mass of the occupation measure may escape to infinity.

The moments of the occupation measure μ is defined, as shown in section 2.3, by integration of monomials with respect to μ . The α -th moment of μ over the support X is given by

$$y_\alpha = \int_X x^\alpha d\mu, \quad \forall \alpha \in \mathbb{N}^n.$$

Moments of occupation measures can be also written as integration over time as

$$y_\alpha = \int_0^T [x(t)]^\alpha dt, \quad \forall \alpha \in \mathbb{N}^n.$$

where $x(t)$ denotes the solution of the ODE starting at x_0 . This follows from the definition of μ . Therefore, if we can find the moments and handle the representation conditions, solving the moments gives the solution of the ODE.

The question now is how to generate the moments of occupation measures corresponding to some vector field. We do this by looking at the time derivative of a continuously differentiable test function, call it v , supported on X . For the simple autonomous system given in (3.3), and some arbitrary $v \in \mathcal{C}^1(X)$, it holds

$$\dot{v} = \nabla v \cdot \dot{x} = \nabla v \cdot f \quad (3.4)$$

where ∇v denotes the gradient of v . Since the solution $x(t)$ of the ODE is absolutely continuous, we can use the fundamental theory of calculus to integrate (3.4) and get the following linear relation

$$\int_X \nabla v \cdot f d\mu = v(x_T) - v(x_0) \quad \forall v, \quad (3.5)$$

where $x_T = x(T)$ is the final value of the solution. The above relation is valid for all continuously differentiable functions. Assuming that the vector field f is a polynomial mapping, the moments can be easily generated by using monomials as the test functions. Thus, the nonlinear ODE is written as an infinite system of linear equations in the occupation measure space.

Furthermore, it is possible to model both the initial and final condition using probability measures. In this case we are looking at all the possible trajectories starting from X_0 and ending in X_T . We then get similar linear relations in the measures space. If the initial measure μ_0 is a Dirac measure supported on $\{x_0\}$, and the terminal measure μ_T is a Dirac measure supported on $\{x_T\}$, we get the same relation (3.5). The relation rewritten in terms of μ_0 and μ_T is called the weak or variational formulation, and reads

$$\int_X \nabla v \cdot f d\mu = \int_{X_T} v d\mu_T - \int_{X_0} v d\mu_0 \quad \forall v. \quad (3.6)$$

This equation gives a weak reformulation of the nonlinear ODE in the measure space. In general, the approach is capable of modeling different initial and final conditions by μ_0 and μ_T , which may be considered as unknowns. This relaxes the ODE to consider a set of initial conditions and a set of final conditions with corresponding probability measures.

3.1.2 Duality between measures and bounded continuous functions

Let $\mathcal{M}(X)$ denote the space of signed Borel measures on X with elements denoted by μ . Furthermore let $\mathcal{C}(X)$ be the space of bounded continuous functions on X , equipped with the supremum norm. Since we defined the set X to be a compact subset of the Euclidean spaces, $\mathcal{M}(X) \simeq \mathcal{C}(X)^*$ [17]. In other terms, the space $\mathcal{M}(X)$ is the dual space of $\mathcal{C}(X)$ with the following duality bracket:

$$\langle v, \mu \rangle = \int_X v d\mu, \quad \forall (v, \mu) \in \mathcal{C}(X) \times \mathcal{M}(X). \quad (3.7)$$

The duality bracket is useful in producing an injective linear map between the space of linear operators on \mathcal{M} and the space of linear operators on \mathcal{C} . For example, assume any compact subset $Y \subset \mathbb{R}^n$, and consider the following linear map

$$\mathcal{L} : \mathcal{C}(X) \rightarrow \mathcal{C}(Y).$$

The adjoint map on measures

$$\mathcal{L}^* : \mathcal{M}(Y) \rightarrow \mathcal{M}(X)$$

is defined using the duality bracket

$$\langle v, \mathcal{L}^*(\mu) \rangle = \langle \mathcal{L}(v), \mu \rangle. \quad (3.8)$$

The duality between Borel measures and bounded continuous functions is captured by convex optimization duality. For example if we consider a primal linear program on (nonnegative) measures

$$\begin{aligned} p^* &= \inf_{\mu \in \mathcal{M}(X)} && \langle c, \mathcal{L}^*(\mu) \rangle \\ \text{s.t.} &&& \mathcal{L}^*(\mu) = b \\ &&& \mu \geq 0 \end{aligned} \quad (3.9)$$

the dual linear program will be on (nonnegative) bounded continuous functions

$$\begin{aligned} d^* &= \sup_{v \in \mathcal{C}(X)} && \langle v, b \rangle \\ \text{s.t.} &&& z = c - \mathcal{L}(v) \\ &&& z \succeq 0. \end{aligned} \quad (3.10)$$

Using duality brackets we can rigorously derive a linear differential equation relating μ , μ_0 and μ_T of the simple nonlinear system (3.3) that describes the transport of the occupation measure. Define the linear map $\mathcal{L} : \mathcal{C}^1(X) \rightarrow \mathcal{C}(X)$ by

$$\mathcal{L}(v) := -\nabla v \cdot f.$$

To identify the adjoint map $\mathcal{L}^*(\mu)$ we should use the duality bracket defined above

$$\begin{aligned} \langle v, \mathcal{L}^*(\mu) \rangle &= \langle \mathcal{L}(v), \mu \rangle \\ &= \langle -\nabla v \cdot f, \mu \rangle \\ &= - \sum_{k=1}^n \int_X \frac{\partial v}{\partial x_k} f_k d\mu \\ &= \sum_{k=1}^n \int_X v \frac{\partial}{\partial x_k} (f_k \mu) = \langle v, \nabla \cdot (f\mu) \rangle \end{aligned} \quad (3.11)$$

this shows that the adjoint operator on measures is

$$\mathcal{L}^*(\mu) = \nabla \cdot (f\mu)$$

where $\nabla \cdot (f\mu)$ denotes the divergence of measure $f\mu$ in the sense of distributions. From the weak formulation (3.6) we get

$$\int_X \mathcal{L}(v) d\mu = \int_{X_0} v d\mu_0 - \int_{X_T} v d\mu_T \quad \forall v,$$

and it follows from (3.8) that

$$\langle \mathcal{L}(v), \mu \rangle = \langle v, \mu_0 \rangle - \langle v, \mu_T \rangle = \langle v, \mathcal{L}^*(\mu) \rangle.$$

We can now eliminate the functions v , and get the following linear differential equation in the measures space:

$$\nabla \cdot (f\mu) = \mu_0 - \mu_T. \quad (3.12)$$

This linear PDE is called the measure transport equation or Liouville's continuity equation, and it corresponds to the nonlinear ODE with initial and terminal probability measures μ_0 and μ_T .

3.1.3 The primal formulation

We now consider the piecewise-affine optimal control problem (3.1). We assume that the control trajectory $u(t)$ is admissible such that all the constraints of the OCP are respected. Accordingly, we define the local occupation measure associated with the cell X_i to be

$$\mu_i(X_i \times U) = \int_0^T \mathbb{I}_{X_i \times U}(x(t), u(t)) dt. \quad (3.13)$$

It encodes the trajectories in the sense that it measures the total time spent by the trajectories $x(t)$, and $u(t)$ in the admissible set $X_i \times U$. Furthermore, define the global state-action occupation measure for the trajectory $(x(t), u(t))$ as

$$\mu(X \times U) = \int_0^T \mathbb{I}_{X \times U}(x(t), u(t)) dt = \sum_{i=1}^r \mu_i(X_i \times U). \quad (3.14)$$

The indicator function $\mathbb{I}_{X \times U}(x(t), u(t)) = 1$ through the interval $[0, T]$ and the mass of μ is given by the terminal time T . The initial and terminal occupation measures are probability measures supported on X_0 and X_T respectively.

The objective function of problem (3.1) can be written in terms of these measures to get the following linear cost

$$J(x_0, u(t)) = \sum_{i=1}^r \int_{X_i \times U} L_i d\mu_i + \int_{X_T} L_T d\mu_T. \quad (3.15)$$

If the Lagrangian is the same for all the cells, say L , the performance measure can be written in terms of the global state-action occupation measure as

$$J(x_0, u(t)) = \int_{X \times U} L d\mu + \int_{X_T} L_T d\mu_T. \quad (3.16)$$

With duality brackets, it reads

$$J(x_0, u(t)) = \sum_{i=1}^r \langle L_i, \mu_i \rangle + \langle L_T, \mu_T \rangle. \quad (3.17)$$

Our next step is to determine the measure transport equation that encodes the piecewise-affine dynamics in the measure space. To do that, we define a compactly supported global test function $v \in \mathcal{C}^1(X)$. Then for $i = 1, \dots, r$ we define a linear map $F_i : \mathcal{C}^1(X_i) \rightarrow \mathcal{C}(X_i \times U)$

$$F_i(v) := -\dot{v} = -\nabla v \cdot (A_i x + a_i + B_i u). \quad (3.18)$$

Integration gives the following

$$\begin{aligned} \int_0^T dv &= - \sum_{i=1}^r \int_{X_i \times U} F_i(v) d\mu_i \\ &= \int_{X_T} v d\mu_T - \int_{X_0} v d\mu_0. \end{aligned} \quad (3.19)$$

Equivalently, we can write

$$\sum_{i=1}^r \langle F_i(v), \mu_i \rangle + \langle v, \mu_T \rangle = \langle v, \mu_0 \rangle \quad \forall v \in \mathcal{C}^1(X). \quad (3.20)$$

Now define the following $(r+1)$ -tuple, in which we gather all the local occupation measures as the first r elements, and the terminal occupation measure as the last element

$$\nu := (\mu_1, \dots, \mu_r, \mu_T).$$

The piecewise-affine optimal control problem (3.1) is equivalent to the following infinite-dimensional linear optimization problem over occupation measures:

$$\begin{aligned} p^* &= \inf_{\nu} \sum_{i=1}^r \langle L_i, \mu_i \rangle + \langle L_T, \mu_T \rangle \\ \text{s.t.} \quad &\sum_{i=1}^r \langle F_i(v), \mu_i \rangle + \langle v, \mu_T \rangle = \langle v, \mu_0 \rangle \quad \forall v \in \mathcal{C}^1(X). \end{aligned} \quad (3.21)$$

Furthermore, define the linear mapping $\mathcal{L} : \mathcal{C}^1(X) \rightarrow \prod_{i=1}^r \mathcal{C}(X_i) \times \mathcal{C}(X_T)$, then rewrite the above relation as

$$\begin{aligned} \langle (F_1(v), \dots, F_r(v), v), \nu \rangle &= \langle \mathcal{L}(v), \nu \rangle \\ &= \langle v, \mathcal{L}^*(\nu) \rangle = \langle v, \mu_0 \rangle, \quad \forall v \in \mathcal{C}^1(X). \end{aligned} \quad (3.22)$$

This defines the adjoint map $\mathcal{L}^* : \prod_{i=1}^r \mathcal{M}(X_i) \times \mathcal{M}(X_T) \rightarrow \mathcal{C}^*(X)$. The measure transport equation is then given by

$$\mathcal{L}^*(\nu) = \mu_0. \quad (3.23)$$

Equation (3.11) then yields

$$\mathcal{L}^*(\nu) = \sum_{i=1}^r \nabla \cdot (f_i \mu_i) - \mu_T \quad (3.24)$$

with the symbol f_i denoting the dynamics in the cell with index $i = 1, \dots, r$.

Define the tuple $c := (L_1, \dots, L_r, L_T)$, and associate the piecewise-affine optimal control problem to the following infinite-dimensional linear program

$$\begin{aligned} p^* &= \inf_{\nu} \quad \langle c, \nu \rangle \\ \text{s.t.} \quad &\mathcal{L}^*(\nu) = \mu_0 \\ &\nu \succeq 0. \end{aligned} \quad (3.25)$$

This is the primal formulation of the optimal control problem in terms of occupation measures of the trajectory $(x(t), u(t))$. The nonlinear nonconvex continuous-time optimal control problem of PWA system is therefore reformulated as an infinite-dimensional LP in the measure space.

It was shown in [18] that if the associated abstract infinite-dimensional LP is feasible, the duality gap vanishes and the optimal value gives a lower bound on the optimal cost of the original OCP. Let d^* be the optimal value of the dual problem (defined in the next subsection). We can find an admissible control trajectory such that $\nu \succeq 0$, and in this case the infimum is attained such that $p^* = d^* \leq v^*(x_0)$. Furthermore, one of the main results in [27] shows that under some convexity condition, the original optimal control problem is solvable and that the optimal value of the associated infinite-dimensional linear program coincides with the optimal cost, and we have

$$p^* = d^* = v^*(x_0). \quad (3.26)$$

Note that LPs (3.21) and (3.25) are equivalent. To proceed numerically we restrict the continuously differentiable functions to be polynomial functions of the state. In other words, we consider $v \in \mathbb{R}[x] \subset \mathcal{C}^1(X)$. By this restriction, we obtain an instance of the GMP (2.14), i.e. an infinite-dimensional linear program over moments sequences corresponding to the occupation measures. As was shown in the previous chapter in subsection 2.3.3, the GMP can be approached using a converging hierarchy of LMIs by fixing the order of the moment and localizing matrices.

To write the semidefinite relaxation of the primal infinite-dimensional LP, let $y_i = (y_{i_\alpha})$, $\alpha \in \mathbb{N}^n \times \mathbb{N}^m$ be the moments sequence corresponding to the local occupation measure μ_i , $i = 1, \dots, r$. Moreover, let $y_0 = (y_{0_\beta})$ and $y_T = (y_{T_\beta})$ with $\beta \in \mathbb{N}^n$ be the moment sequences corresponding to μ_0 and μ_T respectively.

The LMI relaxation of order d of the GMP instance (3.25) can then be formulated by

taking test functions $v = x^\alpha$ with $\alpha \in \mathbb{N}^n$, such that $\deg v = 2d$ as follows

$$\begin{aligned}
p_d^* = & \inf_{y_0, y_1, \dots, y_r, y_T} \sum_{i=1}^r \ell_{y_i}(L_i) + \ell_{y_T}(L_T) \\
\text{s.t.} & \sum_{i=1}^r \ell_{y_i}(F_i(v)) = \ell_{y_T}(v) - \ell_{y_0}(v), \\
& M_d(y_i) \succeq 0, \quad \forall i = 1, \dots, r \\
& M_d(p_{i,k} y_i) \succeq 0, \quad \forall i = 1, \dots, r \quad \forall k = 1, \dots, m_g, \\
& M_d(y_0) \succeq 0, \quad M_d(p_{0,k} y_0) \succeq 0, \quad \forall k = 1, \dots, m_0, \\
& M_d(y_T) \succeq 0, \quad M_d(p_{T,k} y_T) \succeq 0, \quad \forall k = 1, \dots, m_T.
\end{aligned} \tag{3.27}$$

As discussed in the previous chapter, the minimum relaxation order has to allow the enumeration of all the moments appearing in the objective function and the linear equality constraint.

3.1.4 The dual formulation

As mentioned earlier in this chapter, the duality between finite measures and compactly supported bounded continuous functions is captured by convex analysis duality, see (3.9) and (3.10). The dual of the LP (3.25) is thus formulated over the space of positive bounded continuously differentiable functions as follows

$$\begin{aligned}
d^* = & \sup_{v \in \mathcal{C}^1(X)} \langle v, \mu_0 \rangle \\
\text{s.t.} & z = c - \mathcal{L}(v) \\
& z \geq 0
\end{aligned} \tag{3.28}$$

The dual LP can be written in more explicit form to reveal the structure of the linear constraints. We can equivalently write

$$\begin{aligned}
d^* = & \sup_{v \in \mathcal{C}^1(X)} \langle v, \mu_0 \rangle \\
\text{s.t.} & L_i + F_i(v) \geq 0, \quad \forall i = 1, \dots, r, \\
& L_T - v(x_T) \geq 0
\end{aligned} \tag{3.29}$$

and more explicitly

$$\begin{aligned}
d^* = & \sup_{v \in \mathcal{C}^1(X)} \int_{X_0} v d\mu_0 \\
\text{s.t.} & \nabla v(x) \cdot f_i + L_i(x, u) \geq 0, \quad \forall (x, u) \in X_i \times U, \quad \forall i = 1, \dots, r \\
& L_T - v(x_T) \geq 0, \quad \forall x \in X_T.
\end{aligned} \tag{3.30}$$

We note that any feasible solution of the above SDP is actually a global smooth subsolution of the HJB equations. By conic complementarity, along optimal trajectory (x^*, u^*) , it holds

$$\langle z^*, \nu^* \rangle = 0. \tag{3.31}$$

Therefore, for the optimal dual function v^* , the following holds:

$$\nabla v^*(x^*) \cdot f_i + L_i(x^*, u^*) = 0, \quad \forall (x^*, u^*) \in X_i \times U, \quad \forall i = 1, \dots, r \tag{3.32}$$

and in addition,

$$v^*(x^*(T)) = L_T(x^*(T)). \tag{3.33}$$

This is an important result. It shows the following:

1. With a careful look, we can easily identify what we have in (3.32) to be the HJB PDE of the PWA optimal control problem satisfied along optimal trajectories, with the terminal conditions given by (3.33). The HJB PDE is defined by (2.5), and the optimal control Hamiltonian is defined by (2.6). The only difference here is that we have a system of control Hamiltonians defined locally in each cell X_i of the state space. The control Hamiltonian in cell X_i is defined using the Lagrangian L_i and the local corresponding dynamics f_i .
2. The optimal dual function $v^*(x)$ is equivalent to the value function of the optimal control problem, hence the notation. In other words, the continuously differentiable maximizer function $v^*(x)$ of the dual infinite-dimensional LP in equation (3.28) solves, globally, the HJB equation of the PWA optimal control problem along optimal trajectories.

The dual convex relaxation, dual of LMI (3.27), is formulated over positive polynomials. Putinar's Positivstellensatz is used to enforce positiveness. Therefore, the unknown dual variables are the coefficients of the polynomial v and several SOS polynomials that deal with the polynomial positivity conditions of the constraints. The dual program can then be written as follows:

$$\begin{aligned}
 d_d^* = & \sup_{v_d, s} \int_{X_0} v_d d\mu_0 \\
 \text{s.t. } & L_i + F_i(v_d) = s_{i,0} + \sum_{k=1}^{m_i} p_{i,k} s_{i,k} \quad \forall (x, u) \in X_i \times U, \quad \forall i = 1, \dots, r \quad (3.34) \\
 & L_T - v_d(x) = s_{T,0} + \sum_{k=1}^{m_T} p_{T,k} s_{T,k} \quad \forall x \in X_T
 \end{aligned}$$

in which the degree of v_d is d . The polynomials $s_{i,0}$, $s_{i,k}$, $s_{T,0}$ and $s_{T,k}$ are positive. They are Putinar's SOS representations of the constraint polynomials [20]. The polynomials $p_{i,k}$ define the set $X_i \times U$ and the polynomials $p_{T,k}$ define the set X_T , see eq. (3.2).

3.2 Suboptimal control synthesis

Assume that the analytical value function $v^*(x)$ of a general optimal control problem is available by solving the HJB PDE. The optimal feedback control function $k^*(x(t))$ can then be selected such that it generates an admissible optimal control trajectory $u^*(t)$ that satisfies the optimal necessary and sufficient conditions presented in subsection 2.2.2. The resulting optimal feedback function $k^*(\cdot)$ generates the admissible optimal control trajectory starting from any initial value x_0 . Therefore we get the solution of the OCP as a feedback strategy, namely, if the system is at state x , the control is adjusted to $k^*(x)$. This approach has two difficulties. Firstly, the assumption that the analytical value function is continuously differentiable is not always true. Secondly, even if the value function is smooth there would be in general no continuous optimal state feedback law that satisfies the optimality conditions for every state.

It is well-known from Brockett's existence theorem (1983) that if a dynamical control system with some vector field $f(x, u)$ admits a continuous stabilizing feedback (taking the origin as equilibrium point), then for every $\delta > 0$, the set $f(B(0, \delta), U)$ is a neighborhood of the origin [5], where $B(0, \delta)$ is a ball of radius δ with a center at 0. This is a necessary

condition which requires the set U to contain a neighborhood of 0, so that near 0 there are no constraints on the control and all state values in this neighborhood are admissible. In the simplest problems this may fail. One famous example is the nonholonomic integrator [8]. More recently it was shown that asymptotic controllability of a system is necessary and sufficient for state feedback stabilization without insisting on the continuity of the feedback. The synthesis of such discontinuous feedbacks was described in [7], together with a definition of a new solution concept for an ODE with discontinuous dynamics, namely the sample-and-hold implementation. The concept actually introduces a reasonable physical meaning not available for other concepts. In the following few lines the sample-and-hold implementation of a stabilizing feedback is briefly described.

Define $\pi = \{t_i\}_{i \geq 0}$ to be a partition of the time interval $[0, T]$. The number of the subintervals is related to the robustness properties of the feedback law and depends on the specific case of each controlled system. For a given state $x_i = x(t_i)$, the solution $x(t)$ is defined in a sample-and-hold fashion. A control value $k(x_i) = u_i \in U$ is chosen such that the gradient with respect to time of an associated control Lyapunov function is negative (steepest descent feedback). Then, through the subinterval from t_i to t_{i+1} the classical solution of the locally Lipschitz ODE

$$\dot{x}(t) = f(x(t), k(x_i)), \quad x(t_i) = x_i, \quad t_i \leq t \leq t_{i+1}$$

is computed. The value $x(t_i)$ is the state at the end point of the previous partition. Doing that, the existence and uniqueness of the solution $x(t)$ is guaranteed and blow-up of trajectories cannot occur. For more details about the technique refer to [7]. This strategy can also be used to synthesize suboptimal controllers. This can be done by using the “optimal” control Lyapunov function, namely the value function corresponding to the OCP. This leads to the least cost stabilization. It turns out that this concept is very convenient for PWA dynamical systems where the dynamics and control Hamiltonian differ from one cell to another.

Assuming that a polynomial approximation of the value function for a given PWA continuous system is available by solving the relaxed dual LMI, the proposed suboptimal feedback policy $[0, T] \times \mathbb{R}^n \rightarrow U$ is constructed using closed-loop sampling as follows:

1. The time set is partitioned starting from the initial time of the OCP. For the first partition, $t = 0$ and $x = x_0$.
2. The global minimizer u^* of the control Hamiltonian is found at the initial state of the partition by globally solving a static polynomial minimization problem over the set of admissible inputs.
3. Using the constant control u^* , and starting at the initial state of the partition, the ODE $\dot{x} = A_i x + a_i + B_i u^*$ is integrated to generate a near-optimal trajectory. The matrices A_i, B_i, a_i are the matrices of the system in the cell determined by the initial state. The integration is terminated when the trajectory arrives at a boundary of a cell, or at the end of the time partition.
4. The initial states of the next partition is set to the value of x at the end point of step 3. A new control u^* is found by repeating step 2.
5. Stop when we arrive at a neighborhood of the final state x_T .

Generally it is required to have a good approximation of the value function in the neighborhood of the terminal state x_T . To avoid overshoots, the sampling rate should be increased while approaching x_T . For example if the approximating polynomial does not resemble the analytical value function near x_T , the generated trajectory will likely miss the target set and the stopping condition of the algorithm will never be met. The shorter the time partition is,

the closer we are to the optimal trajectory suggested by the solution of the moments problem. This suggests that an upper limit of the step size is required, also not to lose stabilization. If state measurement/estimation error is considered, one may need to introduce a lower bound. For the feedback law to be "relatively robust", the diameter of the partition should be big enough relative to the potential errors in the state. This is especially important if we have a discontinuous control action. The effect of the errors (if considered) on optimality can be significant at the cells boundaries.

The delivered feedback is a closed-loop sampled feedback such that the suboptimal trajectory $x(t)$ generated corresponds to a piecewise constant open loop control. As a guarantee of suboptimality, we calculate and record the cost along the generated trajectory $x(t)$ and compare it to the value of the approximated value function or to the analytic value function if available. The time to reach the terminal state can also be used as a guarantee of having near optimal trajectories. It is compared to the time obtained by the measure problem. In the next chapter, the proposed strategy is applied on several optimal control problems assuming a constant sampling rate.

Chapter 4

Illustrative examples

This chapter demonstrates the suboptimal control synthesis algorithm developed in section 3.2. Three illustrative numerical examples with known analytical solutions are considered. Example 4.1 solves a time-optimal control problem of double-integrator system where the optimal solution is known to be a discontinuous feedback. The suboptimal feedback strategy is applied in example 4.2 to solve an optimal control problem of a spring-mass system. Finally, example 4.3 studies an optimal control problem with first order piecewise-affine dynamics.

4.1 Double integrator with state constraints

In this example we demonstrate the suboptimal strategy on a time-optimal control problem with a piecewise continuous optimal feedback. Consider the system

$$\ddot{y} = u, \quad u \in [-1, 1]. \quad (4.1)$$

The evolution of this second order system can be seen as a representation of a frictionless motion of a car along a straight line with position $y \in \mathbb{R}$. Assume that the car has a unit mass. The control u can then be seen as the external force that accelerates the car.

The goal is to solve the problem of steering the system from a given initial condition $x_0 \in \mathbb{R}^2$ to the origin in minimum time. In other words, it is required to park the car at the origin with zero speed in minimum time. It is clear that there exist an optimal control u^* that brings the system to the origin in minimum transfer time. The OCP can be formulated as follows

$$\begin{aligned} \min_{u, T} \quad & \int_0^T dt \\ \text{s.t.} \quad & \dot{x} = f(x, u) = \begin{bmatrix} x_2 \\ u \end{bmatrix} \\ & x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x(T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned} \quad (4.2)$$

The running cost or Lagrangian is $L \equiv 1$. It depends neither on the state nor the control input. In addition, there is no terminal cost. The problem is still well-posed and makes sense, since the termination time is free. It can be approached using variational methods like the PMP. The question now is how to characterize the optimal control $k^*(x)$. We note that the Lagrangian is not a strictly convex function in u , instead it is only convex in u . The control Hamiltonian reads

$$H(x, u, \nabla v^*) = \nabla v^*(x) \cdot f(x, u) + 1, \quad (4.3)$$

and the value function is given by

$$v^*(x) = \begin{cases} -x_2 + \sqrt{2(x_2)^2 - 4x_1}, & \text{when } x \text{ is at the left of } S \\ +x_2 + \sqrt{2(x_2)^2 + 4x_1}, & \text{when } x \text{ is at the right of } S \end{cases} \quad (4.4)$$

where the switching curve S in the state-space has equation $x_2^2 = 2|x_1|$. The value function $v^*(x)$ returns the minimal time required to transfer the system to the origin from a given state x . It is continuous in x , but fails to be differentiable along the switching curve. It turns out that the minimum of H is attained everywhere on the boundary of the control set $U = [-1, 1]$. In other words, the optimal control trajectory $u^*(t)$ cannot take any values other than 1 or -1 , which is known as bang-bang control. Furthermore, it can be shown that such optimal control has at most one switching from one boundary value to the other. The optimal control is characterized using the value function to give

$$k^*(x) = -\text{sign} \frac{\partial v^*(x)}{\partial x_2}. \quad (4.5)$$

The problem is formulated in terms of occupation measures as follows:

$$\begin{aligned} p^* = \min_{\mu} \quad & \int_{X \times U} d\mu \\ \text{s.t.} \quad & \int_{X \times U} \nabla v(x) \cdot f(x, u) d\mu = \int_{X_T} v(x) d\mu_T - \int_{X_0} v(x) d\mu_0, \quad \forall v(x) \in \mathcal{C}^1 \end{aligned} \quad (4.6)$$

where μ is the trajectory occupation measure, μ_T is the terminal occupation measure, and μ_0 is the initial occupation measure. They are supported on $X \times U$, $X_T = \{0\}$, and $X_0 = \{x_0\}$ respectively. Here the state set X is \mathbb{R}^2 . It can be constrained to a relatively large ball such that it contains the optimal trajectories. The feasible controls set is $U = [-1, 1]$.

To proceed numerically, we write the infinite-dimensional LP (4.6) as a GMP which is then solved with relaxation order $d = 7$. The dual to the LMI moment problem yields a polynomial subsolution of the HJB PDE (4.3). A good approximation of the value function is obtained along optimal trajectories. Based on this approximation we apply the suboptimal control policy.

The level sets of the function $v^*(x) - v_7(x)$ are shown in figure 4.1 where v_7 is a 14th degree approximating polynomial obtained at $d = 7$. It shows that, along the optimal trajectory, the polynomial v_7 stays very close to the value function $v^*(x)$ such that the difference between both stays around zero. This is a result of having $v^*(x(0)) - v_7(x(0))$ close to zero.

The results show that the suboptimal strategy succeeds in providing a stabilizing suboptimal discontinuous feedback that drives the system to the origin in a time almost equal to the optimal minimum time given by the value function. A close look at the generated control trajectory by the suboptimal feedback control $k_7(x)$ indicates that control switches between -1 , 0 and 1 .

In the rest of the example, the suboptimal strategy is applied to the same system after introducing some state constraints. Consider the case where the state x_2 is constrained such that $x_2(t) \geq -1$, $\forall t$. This can be seen as adding a constraint on the reverse velocity of the car modeled by the double integrator. How will the optimal control force look like? It is expected that the acceleration can now assume an additional value besides 1 , -1 , which is 0 . This is intuitive, since once the system reach a velocity of -1 the acceleration is set to zero. The OCP formulation is the same with $L \equiv 1$, and no terminal cost. The value function considering the state constraint is available and given by

$$v^*(x) = \begin{cases} x_2^2/2 + x_1 + x_2 + 1, & \text{when } x_1 \geq 1 - x_2^2/2 \\ 2\sqrt{x_2^2/2 + x_1} + x_2, & \text{when } x_1 \leq 1 - x_2^2/2 \text{ and } x_1 \geq -x_2^2/2 \text{ sign } x_2 \\ 2\sqrt{x_2^2/2 - x_1} - x_2, & \text{when } x_1 < -x_2^2/2 \text{ sign } x_2 \end{cases} \quad (4.7)$$

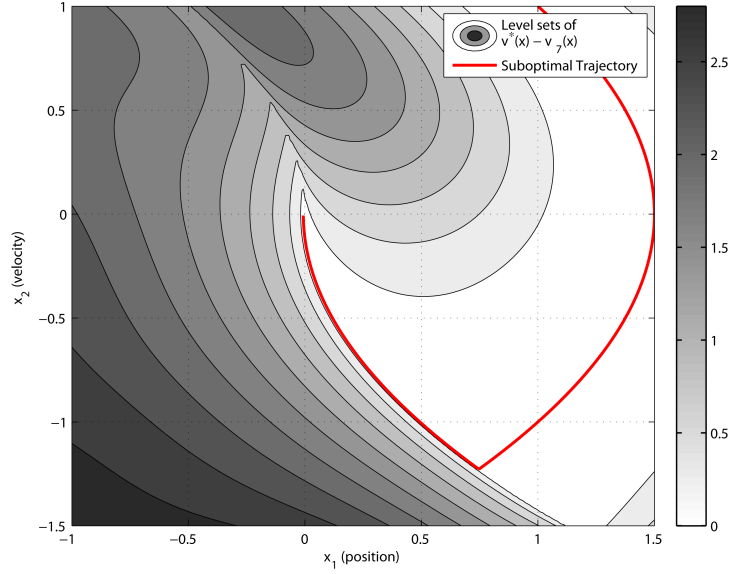


Figure 4.1: Double integrator. Level sets of $v^*(x) - v_7(x)$ and suboptimal trajectory starting from $x_1(0) = x_2(0) = 1$.

The weak formulation of the OCP will be the same, as in (4.6), with the only change in the support of the trajectory occupation measure. The trajectory occupation measure is now supported on $X \times U$ where $X = \{x \in \mathbb{R}^2 \mid x_2 \geq -1\}$ and $U = \{u \in \mathbb{R} \mid u^2 \leq 1\}$. This shows how easy it is to consider state constraints using measures. Figure 4.2 depicts the function $v^*(x) - v_{10}(x)$. The suboptimal trajectory stays in the areas where the difference is close to zero. The difference $v^*(x) - v_{10}(x)$ is large in the regions where the state constraint is violated. The level sets of the function $v^*(x) - v_{10}(x)$ are shown in figure 4.3 together with the generated suboptimal trajectory for $d = 10$.

Again, we see that along the optimal trajectory the difference $v^*(x) - v_{10}(x)$ stays around zero. The suboptimal strategy succeeds in providing a stabilizing suboptimal discontinuous feedback that respects both the state and input constraints.

4.2 Mass spring system

Consider the nonlinear second-order optimal control problem of a mass-spring system:

$$\begin{aligned}
 v^*(x_0) = & \min_{u, T} \int_0^T (4x_2^2 + u^2) dt \\
 \text{s.t.} \quad & \dot{x} = f(x, u) = \begin{bmatrix} x_2 \\ -x_1^3 + u \end{bmatrix} \\
 & x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad x(T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
 \end{aligned} \tag{4.8}$$

This OCP belongs to the class of second order nonlinear problems for which the cost is quadratic and the dynamics are affine in input. The goal here is to use the polynomial approximation of the value function obtained by occupation measure to synthesize a suboptimal state feedback controller for a second order system. The OCP at hand has a known closed form solution found by inverse optimality methods. One can determine explicitly the optimal

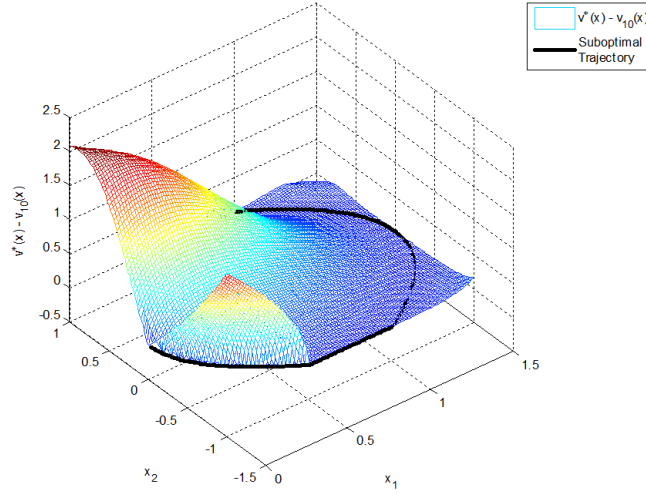


Figure 4.2: Double integrator with state constraints. The function $v^*(x) - v_{10}(x)$, and suboptimal trajectory starting from $x_1(0) = x_2(0) = 1$.

state feedback controller by solving the HJB equation for the optimal policy $k^*(x)$ which can then be used to find the value function.

The analytical optimal feedback control is

$$k^*(x) = -2x_2 \quad (4.9)$$

with value function

$$v^*(x) = 2(x_2^2 + 0.5x_1^4). \quad (4.10)$$

The vector field of the OCP is a multivariate polynomial of the states x and the input u . It is also the case for the Lagrangian. The input $u : [0, T] \mapsto \mathbb{R}$ is assumed to be a bounded measurable function. The problem can thus be formulated in terms of occupation measures. Let μ to be the trajectory occupation measure with mass T , supported on $X \times U$, where X is the state-space and U is the input-space. The OCP comes with neither states constraints nor input constraints. Consequently, it might be necessary to introduce constraints on both the states and the input to satisfy the convergence conditions of the LP and to avoid blow up of trajectories. The initial and final states are modeled with measures μ_0 and μ_T respectively, where the initial measure is set to a given Dirac measure. The OCP can then be written as an infinite-dimensional LP in the measure space

$$\begin{aligned} p^* = & \min_{\mu} \int_{X \times U} (4x_2^2 + u^2) d\mu \\ \text{s.t.} & \int_{X \times U} \nabla v(x) \cdot f(x, u) d\mu = \int_{X_T} v(x) d\mu_T - \int_{X_0} v(x) d\mu_0, \quad \forall v(x) \in \mathcal{C}^1 \end{aligned} \quad (4.11)$$

where $X_T = \{x(T)\}$, $X_0 = \{x(0)\}$ as given in (4.8). This infinite-dimensional LP on measures can be written as a GMP which is then solved numerically.

Along optimal trajectories, a good polynomial approximation of the optimal value function is obtained by solving the dual LMI relaxation of the infinite-dimensional LP (4.11). Figures 4.4 and 4.5 shows the approximating value function obtained at a relaxation order $d = 6$. Based on this polynomial approximation, a suboptimal state feedback controller is

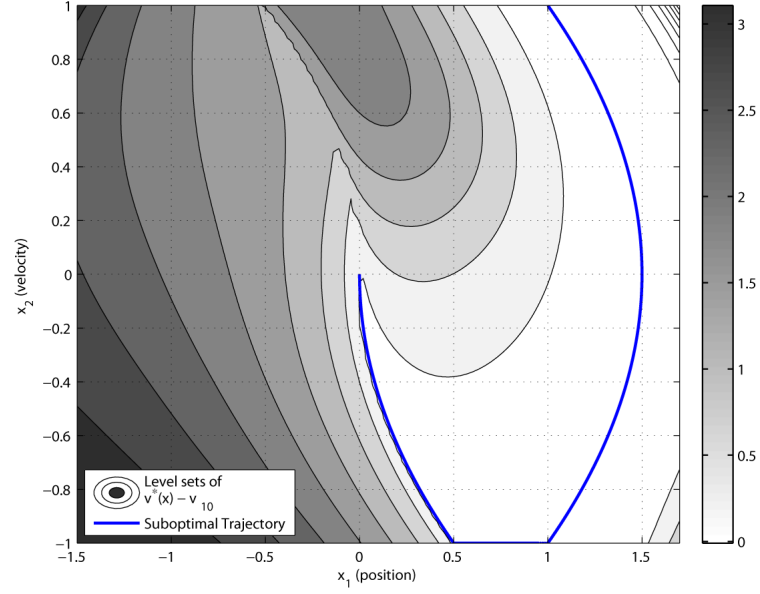


Figure 4.3: Double integrator with state constraints. The level sets of $v^*(x) - v_{10}(x)$, and suboptimal trajectory starting from $x_1(0) = x_2(0) = 1$.

synthesized by using the algorithm in section 3.2. In figure 4.6, the generated suboptimal trajectory is shown. The same figure depicts the level sets of the function $v^*(x) - v_6(x)$. It is clear that along the optimal trajectory the difference between the optimal value function and the approximating polynomial is almost zero. Guarantees of sub-optimality of the generated trajectory is given in table 4.1.

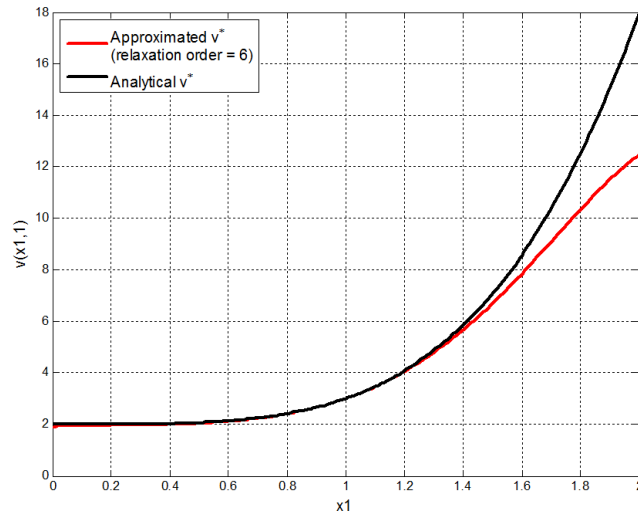


Figure 4.4: Mass spring system. Value function constrained to $x_2 = 1$.

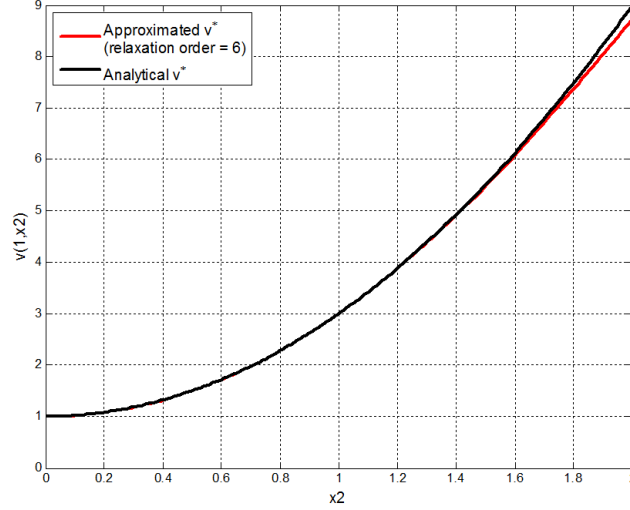


Figure 4.5: Mass spring system. Value function constrained to $x_1 = 1$.

Table 4.1: Mass spring system. Guarantees of sub-optimality in terms of termination time and running cost.

	Measures	Suboptimal trajectory
Termination time:	269.732	320 (at $x_1 = 0.0431, x_2 = -0.0002$)
Running cost:	3.0004	2.9941

4.3 First-order PWA system

We consider a first-order piecewise-affine dynamical system with two cells. The optimal control problem consists of finding a state feedback control to follow admissible control and state trajectories from a given initial state to a given final state while minimizing a specific integral cost functional. The optimization is done over both the control action and time horizon (free time). The OCP is defined as follows:

$$\begin{aligned}
 v^*(x_0) = & \min_{u, T} \int_0^T (2(x-1)^2 + u^2) dt \\
 \text{s.t. } & \dot{x} = \begin{cases} -x + 1 + u & \text{if } x \in X_1 \\ x + 1 + u & \text{if } x \in X_2 \end{cases} \\
 & x(0) = -1, \quad x(T) = +1
 \end{aligned} \tag{4.12}$$

where v^* is the value function, and $x_0 = x(0)$ is the initial state. The control $u(t)$ is assumed to be a bounded measurable function defined on the interval $t \in [0, T]$, and taking its values in \mathbb{R} . The state space is partitioned into two unbounded regions

$$\begin{aligned}
 X &= X_1 \cup X_2, \quad \text{where,} \\
 X_1 &= \{x \in X \mid x \geq 0\}, \\
 X_2 &= \{x \in X \mid x \leq 0\}.
 \end{aligned}$$

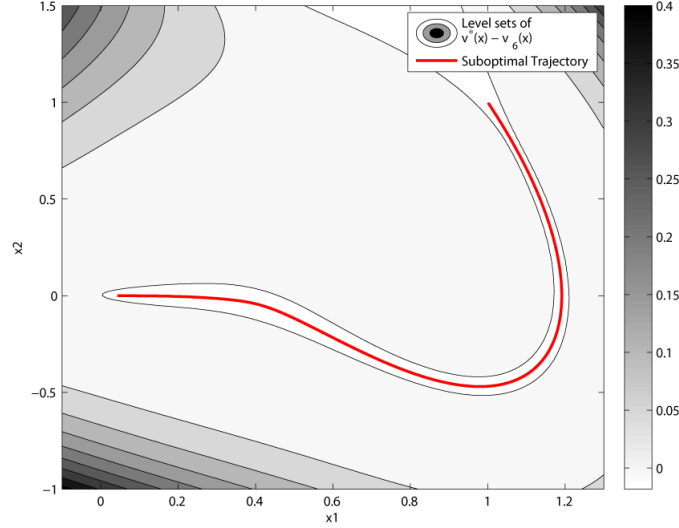


Figure 4.6: Mass spring system. Suboptimal trajectory and level sets of the function $v^*(x) - v_6(x)$.

The control Hamiltonian

$$H(x, u, \nabla v^*) = \begin{cases} \nabla v^*(-x + 1 + u) + (2(x - 1)^2 + u^2) & \text{if } x \in X_1 \\ \nabla v^*(x + 1 + u) + (2(x - 1)^2 + u^2) & \text{if } x \in X_1 \end{cases} \quad (4.13)$$

is used to write the HJB equation. Since the terminal time T is subject to optimization, the control Hamiltonian vanishes along the optimal trajectory.

This gives the HJB PDE

$$\min_{u \in \mathbb{R}} H(x(t), u(t), \nabla v^*(x(t))) = 0, \quad \forall x(t), \quad t \in [0, T]. \quad (4.14)$$

An analytical optimal solution can be obtained by solving the HJB equations corresponding to each algebraic cell. This results in a state feedback control

$$k^*(x) = \begin{cases} (1 - \sqrt{3})(x - 1) & \text{if } x \geq 0 \\ -x - 1 + \sqrt{2(x - 1)^2 + (x + 1)^2} & \text{if } x \leq 0 \end{cases} \quad (4.15)$$

that satisfies the HJB PDE (4.14) for all $t \in [0, T]$ and x . The optimal control trajectory

$$u^*(t) = k^*(x^*(t)) \quad \forall t \in [0, T]$$

is obtained when the optimal state feedback is applied starting at the given initial condition.

The corresponding value function is defined in (4.16). It is found by first finding $\nabla v^*(x)$ from the necessary conditions on the control trajectory to be optimal, $\frac{\partial H}{\partial u} = 0$, and secondly by integration. The partial derivative of the value function with respect to the state is a viscosity solution of the HJB PDE (4.14). The terminal value of the value function is then used to determine the integration constant:

$$v^*(x) = \begin{cases} (-1 + \sqrt{3})(x^2 - 2x) - (1 - \sqrt{3}) & \text{if } x \geq 0 \\ x^2 + 2x + 2.28905 - \frac{1}{6}(6x - 2)\sqrt{(3x^2 - 2x + 3)} \\ \quad - \frac{8}{3\sqrt{3}} \ln(2\sqrt{3}\sqrt{(3x^2 - 2x + 3)} + 6x - 2) & \text{if } x \leq 0. \end{cases} \quad (4.16)$$

The initial and final states are available. Therefore, the initial and terminal occupation measures are Dirac measures. The global occupation measure $\mu \in \mathcal{M}(X \times U)$ is used to encode

the system trajectories. It is defined as a combination of two local occupation measures, one for each cell, as follows

$$\mu = \mu_1 + \mu_2$$

such that the support of μ_1 is $\{(x, u) \mid x \in X_1, u \in U\}$ and the support of μ_2 is $\{(x, u) \mid x \in X_2, u \in U\}$. The OCP can then be formulated as an infinite-dimensional LP in measure space

$$\begin{aligned} p^* = \min_{\mu_1, \mu_2} & \int_{X_1, U} (2(1-x)^2 + u^2) d\mu_1 + \int_{X_2, U} (2(1-x)^2 + u^2) d\mu_2 \\ \text{s.t.} & \int \nabla v \cdot (1-x+u) d\mu_1 + \int \nabla v \cdot (x+1+u) d\mu_1 = \int_{X_T} v d\mu_T - \int_{X_0} v d\mu_0, \quad \forall v \end{aligned} \quad (4.17)$$

where μ_0 and μ_T are the initial and final measures, supported on $X_T = \{+1\}$, $X_0 = \{-1\}$ respectively. The functions v are functions of the state x and belongs to the space of continuously differentiable functions. This infinite-dimensional LP on measures is then written as an instance of GMP as discussed in 2.3.3. The Matlab toolbox GloptiPoly can be used for this task. Here, we are mainly interested in solving the dual formulation on functions.

In the present example, both the state space and the input space are not compact. Hence, the LMI relaxations do not include any localizing constraints; moreover, Putinar's conditions are not sufficient for the convergence of the LMI optimal value. The sufficiency is guaranteed only for measures on compact support. The main concern here is that the mass of the occupation measure is not bounded and might tend to infinity. This is numerically problematic. There are two ways to turn around this. The first is to make the time interval compact by adding a constraint on the mass of the global occupation measure. For example we enforce that the mass of μ is less than a given large positive number. This exploits the fact that the vector field is asymptotically stable. Another way, which is more general, is to constrain X and U to sufficiently large subsets. One possibility is to take X as a large ball centered around 0 (or the initial state x_0). The same can be assumed for the input space U . This is done without any problems as long as the optimal trajectory remains in the constraint set (X, U) . To avoid numerical problems, it is also recommended in all cases to scale down the problem, if possible, to have all the variables inside the unit box. Scaling avoids blow up of the moments sequences for high relaxation orders.

The results obtained below for the OCP (4.12) assume that the global occupation measure is supported on $X \times U$ without introducing any constraints on the spaces.

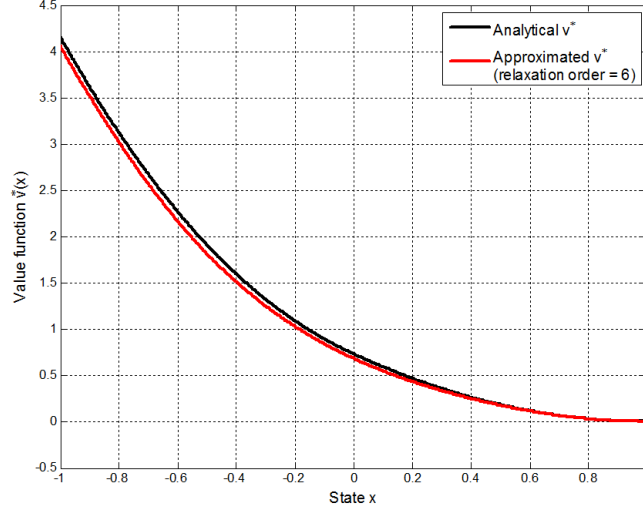
The solution consists of two main steps:

1. Finding a relatively good polynomial approximation of the value function by solving the dual LMI relaxation of the infinite-dimensional LP (4.17).
2. Employing the suboptimal strategy developed in section 3.2 to generate a suboptimal admissible feedback control law based on the smooth approximation of the value function obtained in step 1.

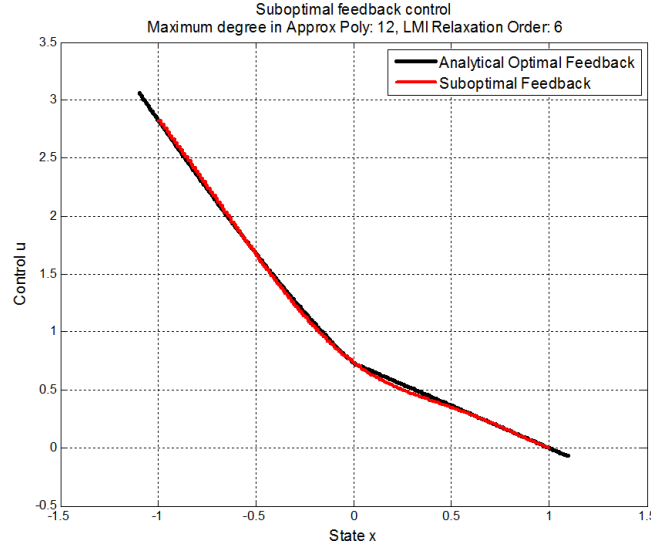
The first step is achieved by solving the relaxed dual LMI on continuous function. Figure 4.7 shows the obtained approximation of the value function with LMI relaxation order $d = 6$.

The obtained approximation is a polynomial of the state x with degree equal to $2d$:

$$\begin{aligned} v_6(x) = & 0.68639 - 1.4639x + 1.2963x^2 - 0.9972x^3 + 0.01507x^4 + 0.71988x^5 \\ & - 0.1142x^6 - 0.39122x^7 + 73.663 \times 10^{-3}x^8 + 91.366 \times 10^{-3}x^9 \\ & - 9.9881 \times 10^{-3}x^{10} - 70.656 \times 10^{-3}x^{11} + 2.5879 \times 10^{-3}x^{12}. \end{aligned} \quad (4.18)$$

Figure 4.7: PWA system. Approximating value function for $d = 6$.

Based on this approximation, we employ the algorithm developed in section 3.2. The resulting suboptimal feedback control is shown in figure 4.8 in comparison with the analytical optimal feedback, $k^*(x)$, calculated using (4.15). It is clear that the algorithm gives a suboptimal feedback close to the optimal. The corresponding suboptimal state trajectory, $x^*(t)$, is shown in figure 4.9.

Figure 4.8: PWA system. Suboptimal feedback for $d = 6$.

In table 4.2 we use two parameters to evaluate the suboptimal strategy, namely the termination time and the running cost. The analytical optimal running cost for the initial state x_0 is given by the value function in (4.16) and used as a reference. The optimal termination time is found by applying the optimal feedback $k^*(x)$ given by (4.15) to the PWA system starting at $x(0)$. The values obtained by solving the relaxed LP on measures are shown in the second column. The termination time is given by the mass of the global occupation measure, and the running cost is the optimal value of the relaxed measures LP. The

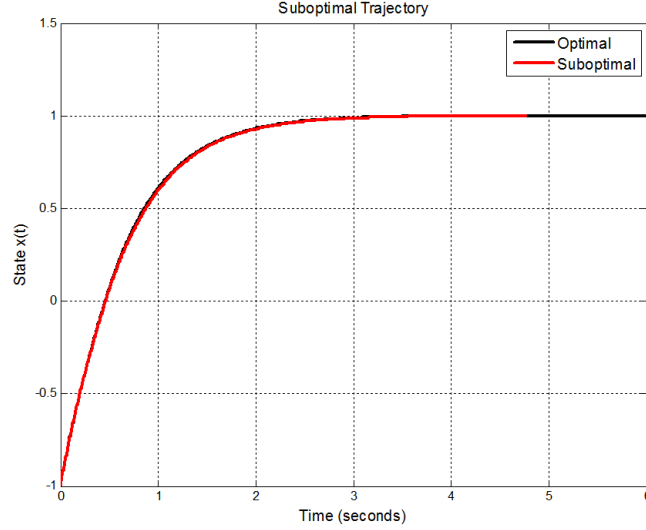
Figure 4.9: PWA system. Suboptimal trajectory for $d = 6$.

Table 4.2: PWA system. Guarantees of sub-optimality in terms of termination time and running cost.

	Analytical Optimal	Measures	Suboptimal trajectory
Termination time:	8.4374	8.1728	4.782
Running cost:	4.157	4.058	4.158

entries of the third column are calculated along the suboptimal trajectories. The termination time is found using the same algorithm used to find the optimal time in the first column. The simulation is terminated when the trajectory arrives at a neighborhood of the final state (note that it approaches x_T asymptotically). The running cost is very close for all cases. In the cases where no analytical solution is available for the OCP, the values returned by the relaxed measures LP can be used as a reference. Such comparison shows how far the suboptimal trajectories are from the optimal ones. It gives a guarantee of sub-optimality.

Chapter 5

Conclusion and open questions

The focus of this project is the synthesis of suboptimal state feedback controllers for continuous-time optimal control problem (OCP) with piecewise-affine (PWA) dynamics and piecewise polynomial cost functions. Both the state constraints and input constraints are considered in a very convenient way and they do not pose additional complexity. The problem is formulated as an abstract infinite-dimensional convex optimization problem over a space of occupation measures which can be seen as an instance of a generalized moment problem (GMP) with polynomial data. Under some convexity assumption, the optimal value of the infinite-dimensional program coincides with the optimal cost of the OCP. Fortunately, assuming all the sets to be compact basic semi-algebraic sets makes the problem approachable. Under such assumptions, recent results from algebraic geometry reduces the problem to a converging hierarchy of semidefinite programming (SDP) or linear matrix inequality (LMI) problems which results in nondecreasing lower bounds on the optimal value. This relaxation allows one to exploit available methods of convex analysis to characterize solutions to such nonlinear nonconvex OCPs. The state and input constraints represents no additional difficulty and are conveniently represented by the support of the occupation measures. The primal LMI is formulated over a space of several local state-action occupation measures corresponding to the state-space partition, in addition to initial and final probability measures. Each local occupation measure encodes the system trajectories over its own support. The initial and final measures can be chosen to be Dirac measures to have the standard OCP of transferring the state from one known point to a desired terminal state. With no additional effort this can be extended to model situations where it is required to optimize over a given initial set and final sets. In this case, by selecting appropriate initial and final probability measures, we optimize over all the possible trajectories starting in the initial set and ending in the terminal set. This transport of measures is described by a linear partial differential equation (PDE) in the measures space linking all occupation measures.

The dual of the original infinite-dimensional program is formulated over the space of bounded continuous functions. Under some compactness-continuity conditions there is no duality gap. By restricting the dual variables to be polynomials and solving the dual program we obtain a polynomial representation of the value function of the OCP along optimal trajectories in terms of upper envelope of subsolutions to a system of Hamilton-Jacobi-Bellman (HJB) partial differential equations (PDEs) corresponding to the state space partition. By fixing the degree of the polynomial, the same dual program can be relaxed and written, using Putinar's Positivstellensatz, as a polynomial sum-of-squares (SOS) program, an LMI problem. The main limitation here is the number of the total variables of the LMI. It is known that the computational complexity of the LMI relaxations increases polynomially with the relaxation order, but the exponent is the number of states and system inputs. In contrast, the complexity increases linearly with the number of cells (local occupation measures). So the number of cells is not actually as critical as the number of states and inputs. In any case,

we must say that the achievable performance of the technique relies on the available SDP solvers. The numerical experiments made in this project were solved using general purpose SDP solvers like SeDuMi. This introduces limitations on the achievable performance which could be removed if a customized SDP solver that exploits the specific structure of the LMIs were used.

As soon as the polynomial approximation of the value function is available, one can systematically generate a suboptimal, yet admissible, feedback control with almost no further assumptions. The suboptimal strategy is based on a closed-loop sampling implementation which is very convenient for PWA systems. The results obtained by applying the strategy depends mainly on the quality of the polynomial approximation of the value function. Due to the fact that the analytical value function is not necessarily continuously differentiable, one may expect that a polynomial will not give a good approximation at several points. It is important to have as good approximation as possible at the initial state at which the trajectory starts to ensure good approximation along the optimal trajectory. In addition, a good approximation in the neighborhood of the terminal state is necessary to avoid missing it. As showed by the numerical examples, the results are positive. The method generates an approximate control signal which is piecewise constant, and near optimal trajectories that respect the given constraints.

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