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DEPARTMENT OF CONTROL ENGINEERING

DIPLOMA THESIS

Control of ill-conditioned systems

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ZADÁNÍ DIPLOMOVÉ PRÁCE

Student: **Bc. Jiří Pich**

Studijní program: Kybernetika a robotika
Obor: Systémy a řízení

Název tématu: **Řízení špatně podmíněných systémů**

Pokyny pro vypracování:

1. Seznamte se s problematikou řízení špatně podmíněných systémů a s prediktivní regulací. Vypracujte rešerši na dané téma, zejména se zaměřte na existující stav poznání v oblasti prediktivní regulace špatně podmíněných systémů.
2. Vypracujte analýzu špatně podmíněných systémů a definujte potřebnou metriku, pomocí které se dá odhadnout míra špatné podmíněnosti systému.
3. Navrhněte vhodné modifikace/doplnění základního algoritmu prediktivního regulátoru takové, aby bylo možné špatně podmíněné systémy touto technologií řídit.
4. Navržené metody ověřte pomocí ilustrativních simulací v Matlabu.

Seznam odborné literatury:

[1] Rossiter J. A., Model-Based Predictive Control, CRC Press, 2003
[2] Maciejowski J., Predictive control with constraints, Prentice Hall, 2001
[3] S. Skogestad, Multivariable Feedback Control, Wiley 2007

Vedoucí: Prof.Ing. Vladimír Havlena, CSc.

Platnost zadání: do konce letního semestru 2011/2012

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V Praze dne 31. 1. 2011

Declaration

I declare that this thesis is my own work and I used only the materials (literature, projects, SW, etc.) quoted in the attached reference list.

In Prague, ... Signature: ....

Acknowledgement

At this point I wish to express my gratitude and thanks to professor Vladimír Havlena for his guidance and helpful comments. I would also like to thank to Mr. Jaroslav Pekař for numerous consultations, that were always helpful and inspiring. Last but not least I would like to thank to my parents for their support and patience.

Abstract

The aim of this thesis is to propose modifications of Model Predictive Controller (MPC) that will improve control of ill-conditioned systems. Based on a background research, an overview of currently used directionality measure and MPC applications for ill-conditioned systems is presented in the thesis. The suitable directionality measures are used for system conditioning analysis. Based on current knowledge and available methods, two extensions of MPC for ill-conditioned systems control are proposed in this thesis.

The first of the presented methods extends the MPC cost function by a term penalizing absolute value of system input. This approach limits the size of the system input absolute value and prevents its unconstrained rise, which may occur if a system output should track a reference which requires a step corresponding to the system small singular values.

The second proposed method extends the cost function by a term penalizing the system input movement in directions corresponding to the small system singular values. This modification prevents undesirably high system inputs in directions where the input has a small impact on the reference tracking abilities.

The proposed MPC modifications are applied to a model of a distillation column to show the reference tracking properties and to check the robust stability of the extended MPC. Robust stability of the system is checked for different uncertainty representations in order to show how the additional criteria influence the frequency characteristics for these different uncertainty representations.

Abstrakt

Cílem této práce je navrhnout modifikace MPC regulátoru, které umožní řízení špatně podmíněných procesů. Práce obsahuje přehled metod používaných ke zjišťování podmíněnosti systémů a metod prediktivního řízení špatně podmíněných systémů. Rešerše na toto téma je uvedena v příloze práce. Na základě tohoto přehledu je navržena metrika určování podmíněnosti systémů a dvě modifikace MPC regulátoru pro řízení špatně podmíněných systémů.

První z navržených metod rozšiřuje ztrátovou funkci MPC o člen penalizující absolutní hodnotu vstupního signálu systému. Tato modifikace omezuje velikost vstupního signálu systému a zabraňuje tím jeho neomezenému růstu, který může nastat jako důsledek snahy regulátoru sledovat referenční signál, jehož dosažení vyžaduje výrazný akční zásah ve směru odpovídajícím malému singulárnímu číslu systému.

Druhá navržená metoda rozšiřuje ztrátovou funkci MPC regulátoru o člen penalizující přírůstek vstupní proměnné ve směru odpovídajícím malým singulárním číslům systému. Tato modifikace zamezuje rychlému růstu vstupní proměnné ve směrech, ve kterých má velikost vstupu systému malý vliv na velikost výstupu systému.

Navržené modifikace prediktivního regulátoru jsou aplikovány na model špatně podmíněné destilační kolony. Na simulacích je ukázána schopnost navržených MPC regulátorů sledovat referenční signál. Různé reprezentace neurčitosti systému jsou použity ke zjištění vlivu navržených úprav regulátoru na stabilitu systému.

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Typesetting and abbreviation

$\mathbf{A}$	matrix
$\mathbf{I}_n$	unit matrix of size n
$\mathbf{x}$	column vector
$\alpha,\ m$	constants
u_k^N	sequence $u_k, \ldots,\ u_N$
$u_k^{\star N}$	sequence of optimal values
$\|\mathbf{w}\|_2,\ \|\mathbf{w}\|$	quadratic norm
$\|\mathbf{w}\|_\infty$	infinity norm

HIIA	Hankel Interaction Index Array
HSV	Hankel Singular Value
LQR	Linear Quadratic Regulator
MIMO	Multi-Input Multi-Output
MPC	Model Predictive Control
MPRC	Model Predictive Range Control
PM	Participation Matrix
RGA	Relative Gain Array
RHP	Right Half-Plane
RS	Robust Stability
SISO	Single-Input Single-Output
SVD	Singular Value Decomposition

Chapter 1

Introduction

Most systems that need to be controlled in practical applications are multi-input multi-output (MIMO) systems. Control of these systems is a challenging task due to the system interactions. One input of a system may influence all the outputs, therefore the system gain can not be easily evaluated as in the single-input single-output (SISO) case. These system interactions may cause fundamental problems for controller design. MIMO system may have a large gain for some input combinations and a very small gain for other input combinations. The more inputs and outputs the system has, the more difficult it is to evaluate the individual impact of a single input on the MIMO system output. Systems with a large difference between the maximum possible gain for some inputs combination and a minimum gain for other inputs combination are called ill-conditioned systems. These systems are particularly difficult to control because it is difficult to drive the system outputs in some directions. Moreover this is not a small group of specific systems, but a large range of systems that are used mainly in chemical and process industry.

Basic controllers are used for control of plants with strong directionality in practical applications. The multivariable systems are diagonalized and controlled with P, PI or PID controllers. Modern control methods are rarely employed in this type of systems. Model Predictive Control (MPC) became a conventional control technique during the last decades. Computing power of up-to-date computers is not a limit for employing this technique anymore. MPC has shown many advantages and many advantageous properties, which makes it a popular choice of control engineers. This contemporary control design technique will be employed to control an ill-conditioned plant in this thesis. Advantage will be taken of the fact that control objectives of MPC cost function may be extended by additional criteria. The aim of this thesis is to find such additional criteria that regard the ill-conditioning of a system and employ MPC into ill-conditioned systems control.

1.1 Objectives of the Thesis

The primary objective of this thesis is to design a Model Predictive Controller that is capable to control ill-conditioned systems. The controller design will be based on information gained from a background study of the topic of ill-conditioned systems and model predictive control. The system conditioning measures will be analyzed in this thesis, and suitable measures will be employed in the MPC design. The proposed MPC will be designed for a model of a distillation column. The reference tracking and stability of the controlled system will be presented and compared to a conventional MPC without the adjustments for ill-conditioned systems.

1.2 Outline of the Thesis

Chapter 1 - Introduction gives an overview of the studied topic and explanation of the proposed control strategy for ill-conditioned systems

Chapter 2 - Introduction to MIMO systems describes the basic properties of MIMO systems and the differences between SISO and MIMO systems. The measures of system directionality and interactions are presented based on a background research on this topic. Control of multivariable plants is overviewed with emphasis on Model Predictive Control.

Chapter 3 - Predictive control of ill-conditioned plants presents some known methods of ill-conditioned plants predictive control. The modified MPC for ill-conditioned plants is proposed in the first part of the chapter. The modified MPC is applied to a model of a distillation column in the second part of the chapter to demonstrate the control system behaviour. Robust stability is verified for different uncertainty representation models.

Chapter 4 - Conclusion brings a summary of the goals and achieved results of the thesis.

Chapter 2

Introduction to MIMO systems

2.1 Basic properties of MIMO systems

In this chapter a brief introduction to MIMO systems will be given. This chapter is based on a textbook by Skogestad et al. [1]. There are some common properties of SISO and MIMO systems, which will be briefly overviewed, but more emphasis will be given on pointing out the differences between these two types of systems. Understanding of the MIMO systems is very important, as in practical control engineering applications most of the systems are multivariable (with multiple inputs and multiple outputs).

2.1.1 System interactions

The fundamental difference between SISO and MIMO systems except the number of inputs and outputs is the fact, that there are interactions between inputs and outputs of MIMO systems. This property is a cause of most of the differences between these two types of systems. A change in input signal of SISO system causes a change in the output signal. This simplicity enables us to control the output variable directly by changing the input variable. On the other hand, a change in one input of MIMO system may influence more than one output variable. Moreover one output variable may be influenced by many input variables, which makes the control system design challenging.

Considering a general MIMO system, a change in input u_1 affects all the outputs y_1, y_2, ...y_n.

In order to control and deal with the MIMO system, it is necessary to have a measure of how much each input affects different outputs. The size of an output variable depends not only on a size of input signals, but also on the combination of individual input sizes, which determines the system input direction. The directional properties of multivariable system may be quantified using Singular Value Decomposition (SVD). More detailed description of the possibilities of how to characterize a MIMO system and its directionality will be given in the rest of this chapter, the different properties and system classifications will be characterized in separate paragraphs.

2.1.2 MIMO system description

MIMO system can be described similarly to a SISO system in terms of transfer function matrix. Basic transfer function model may be given in terms of output equation as $\mathbf{y}(s) = \mathbf{G}(s)\mathbf{u}(s)$. If the system has m inputs and l outputs the output $\mathbf{y}$ is an $l \times 1$ vector, input $\mathbf{u}$ is an $m \times 1$ vector and $\mathbf{G}(s)$ is an $l \times m$ transfer function matrix. A block diagram is a useful representation of the system when transfer function is to be derived. The block diagram of a system with negative feedback is depicted in Figure 2.1.

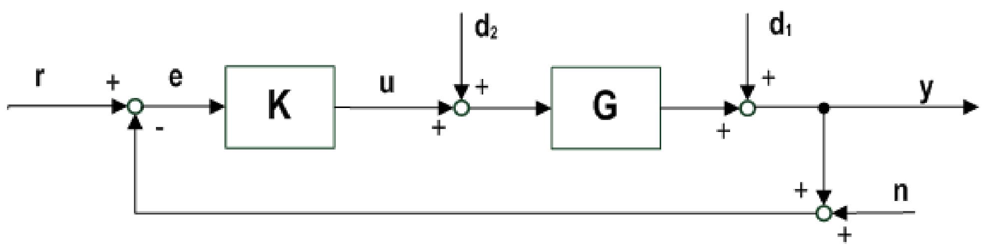

Figure 2.1: Block diagram of a system with negative feedback

In Figure 2.1 $\mathbf{r}$ is a reference vector, $\mathbf{e}$ is tracking error vector, $\mathbf{u}$ is the regulator output vector, $\mathbf{d}_2$ is the system input disturbance vector, $\mathbf{d}_1$ is the system output disturbance vector, $\mathbf{y}$ is the system output vector, and $\mathbf{n}$ is the measurement noise vector.

Block $\mathbf{G}$ in the figure represents a plant transfer function, and block $\mathbf{K}$ represents a controller. When braking the loop at the output of the plant, the transfer function will be $\mathbf{L} = \mathbf{GK}$. At this point, it is possible to define system sensitivity and complementary sensitivity functions. The sensitivity function, sometimes also called output sensitivity is expressed as

$$\mathbf{S} \triangleq (\mathbf{I}+\mathbf{L})^{-1}, \tag{2.1}$$

which is a transfer function from $\mathbf{d}_1$ to $\mathbf{y}$. The complementary sensitivity may be then found as a complement of the sensitivity function to unity. Sometimes it is also called output complementary sensitivity

$$\mathbf{T} \triangleq (\mathbf{I}-\mathbf{S}) = \mathbf{L}(\mathbf{I}+\mathbf{L})^{-1}. \tag{2.2}$$

There are some important relations between the presented system parameters, which should be mentioned at this point:

$$\mathbf{S}+\mathbf{T} = (\mathbf{I}+\mathbf{L})^{-1} + \mathbf{L}(\mathbf{I}+\mathbf{L})^{-1} = \mathbf{I}\ , \tag{2.3}$$

$$\mathbf{G}(\mathbf{I}+\mathbf{KG})^{-1} = (\mathbf{I}+\mathbf{GK})^{-1}\mathbf{G}\ , \tag{2.4}$$

$$\mathbf{GK}(\mathbf{I}+\mathbf{GK})^{-1} = \mathbf{G}(\mathbf{I}+\mathbf{GK})^{-1}\mathbf{K} = (\mathbf{I}+\mathbf{GK})^{-1}\mathbf{GK}\ , \tag{2.5}$$

$$\mathbf{T} = \mathbf{L}(\mathbf{I}+\mathbf{L})^{-1} = (\mathbf{I}+(\mathbf{L})^{-1})^{-1}\ . \tag{2.6}$$

These relations will be used throughout this thesis to obtain different system transfer functions.

2.2 Measures of system directionality and interactions

Evaluating a gain of a MIMO system is not as easy as in the SISO case, where a simple ratio between system output and input magnitude defines the system gain

$$\frac{|y(j\omega)|}{|u(j\omega)|} = \frac{|G(j\omega)u(j\omega)|}{|u(j\omega)|} = |G(j\omega)|\,. \tag{2.7}$$

The SISO gain depends on frequency, but since the system is linear, the system gain is independent of the input magnitude $|u(j\omega)|$. In MIMO systems, the input and output signals are vectors, therefore the equation (2.7) can not be directly applied. In order to evaluate gain of MIMO system, it is necessary to sum up the contribution of all the inputs to the output where we want to evaluate the gain. In addition to size of the resulting vector, the direction of this vector needs to be taken in account. A magnitude of each input and output vector is summed using a norm. In this text the quadratic will be used as it is a standard measure of magnitude. The magnitude of the vector input signal is

$$\|\mathbf{u}(\mathrm{j}\omega)\|_2 = \sqrt{u_1^2 + u_2^2 + \dots}\,, \tag{2.8}$$

similarly the magnitude of the output vector signal is

$$\|\mathbf{y}(\mathrm{j}\omega)\|_2 = \sqrt{y_1^2 + y_2^2 + \dots}\,. \tag{2.9}$$

The system gain is then given as the ratio between these two magnitudes

$$\frac{\|\mathbf{y}(\mathrm{j}\omega)\|_2}{\|\mathbf{u}(\mathrm{j}\omega)\|_2} = \frac{\|\mathbf{G}(j\omega)\mathbf{u}(j\omega)\|_2}{\|\mathbf{u}(j\omega)\|_2} = \frac{\sqrt{y_1^2 + y_2^2 + \dots}}{\sqrt{u_1^2 + u_2^2 + \dots}}\,. \tag{2.10}$$

MIMO system gain is independent of input magnitude, but it is frequency dependent. Comparing to SISO case, there are additional degrees of freedom . The gain depends on the direction of the input vector **u**, where we consider direction as a normalized vector of unit length. The maximum gain as the direction of the input is varied is the maximum singular value of **G**

$$\max_{u\neq 0} \frac{\|\mathbf{Gu}\|_2}{\|\mathbf{u}\|_2} = \max_{\|\mathbf{u}\|_2=1} \|\mathbf{Gu}\|_2 = \overline{\sigma}(\mathbf{G})\,. \tag{2.11}$$

The smallest system gain over all input directions is the minimum singular value

$$\min_{u\neq 0} \frac{\|\mathbf{Gu}\|_2}{\|\mathbf{u}\|_2} = \min_{\|u\|_2=1} \|\mathbf{Gu}\|_2 = \underline{\sigma}(\mathbf{G})\,. \tag{2.12}$$

The ratio between the maximum singular value $\overline{\sigma}(\mathbf{G}(j\omega))$ and the minimum singular value $\underline{\sigma}(\mathbf{G}(j\omega))$ is a good measure of the system directionality. This ratio is called a system condition number

$$\gamma(\mathbf{G}(j\omega)) = \frac{\overline{\sigma}(\mathbf{G}(j\omega))}{\underline{\sigma}(\mathbf{G}(j\omega))}\,. \tag{2.13}$$

The higher the condition number, the more is the plant gain dependent on input direction. The importance and properties of the condition number is discussed in Section 2.2.3. In some cases it is useful to evaluate all the system singular values, not only the maximum and minimum singular value. Using the system matrix **A**, all the system singular values may be calculated as

$$\sigma_i(\mathbf{A}) = \sqrt{\lambda_i(\mathbf{A}^T\mathbf{A})}\,. \tag{2.14}$$

To illustrate the MIMO system directionality it is useful to plot the dependency of the gain on the ratio between input signals and also the dependency of the gain on changing the input direction. A simple example will be given to show these MIMO system properties.

Let's use a 2 x 2 MIMO system [1], with a constant system gain matrix

$$\mathbf{G} = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}.$$

To illustrate the dependency of the gain on the input direction, we will plot the gain $\frac{\|\mathbf{y}\|_2}{\|\mathbf{u}\|_2}$ as a function of the ratio of two inputs $\frac{u_2}{u_1}$, which is taken as an independent variable. From Figure 2.2 it is obvious, that the system gain changes with the ratio between inputs. The maximum and minimum system gains correspond to the maximum and minimum singular value, which may be calculated according to equation (2.11) and (2.12).

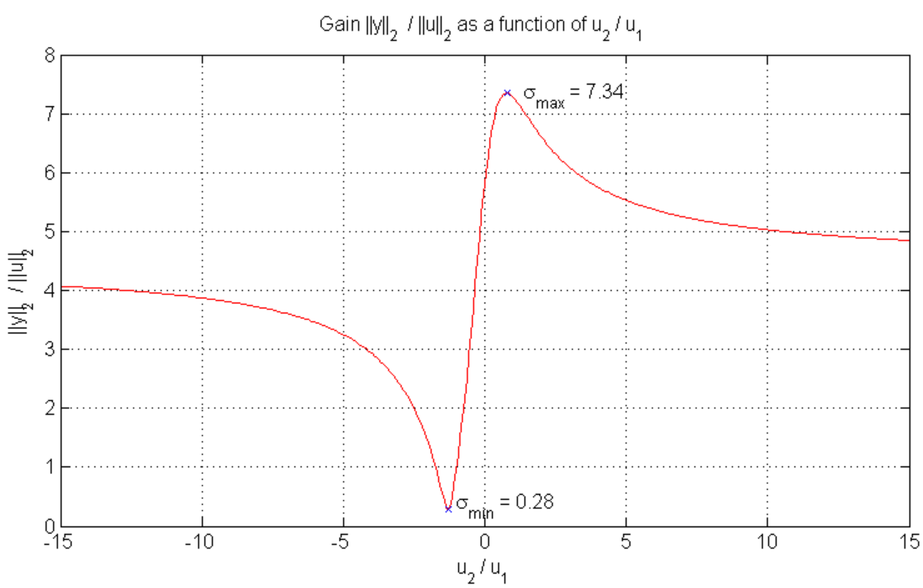

Figure 2.2: Gain plotted as a function of ratio of input signals

The output directions may be plotted considering the input vector of a unit size in all possible directions. Plotting the output y_1 as a function of the output y_2, we receive a very illustrative graph, which is depicted in Figure 2.3. The graph is ellipse shaped. As there is a big difference between the major and minor axis of the ellipse, it is possible to say that the system is ill-conditioned. In the direction of the major axis the gain of the system is very high, which is opposite to the output gain in the direction of the minor axis. This type of plot is useful, as the ratio between maximum and minimum singular value influences the shape of the ellipse. If the ratio is large, the major axis is much larger than the minor axis, and the system gain is very small in some directions, in comparison to other directions. When the shape of the ellipse is conformable to a circle, there are no major directions, and there should be no remarkable problems with system directions when designing a controller. From the plot in Figure 2.3 we can say, that it will be easy to increase or decrease both outputs y_1 and y_2 at the same time (this corresponds to driving the system outputs in the direction σ_{max}), but it will be difficult to increase one output while the other output is being decreased (this corresponds to driving the system outputs in the direction σ_{min}).

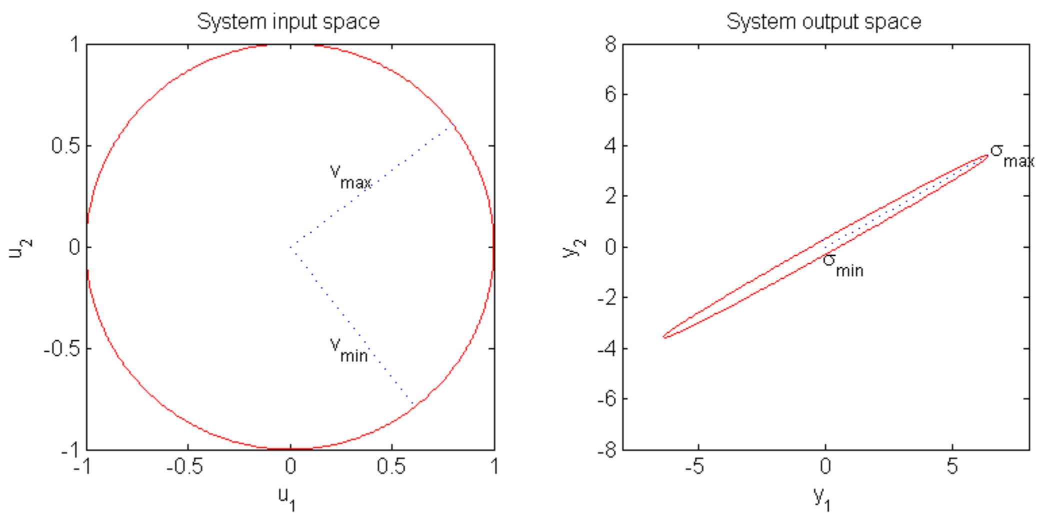

Figure 2.3: System input and output space

According to what has been said about the gain of MIMO system, it should be obvious, that MIMO system eigenvalues can not be simply used as a measure of the system gain. Eigenvalues are a measure of system gain only when the input and output vectors have the same directions, namely the direction of eigenvectors. Therefore there is a need to use other techniques to evaluate the system gain with emphasis on directions.

2.2.1 Singular Value Decomposition

A MIMO system $\mathbf{G}(j\omega)$ with m inputs and l outputs can be decomposed at a fixed frequency ω into a triplet of matrices

$$\mathbf{G} = \mathbf{U}_y \Sigma \mathbf{V}_u^T , \tag{2.15}$$

where Σ is a diagonal $l \times m$ matrix with $k \leq \min\{l, m\}$ non-negative singular values σ_i. Singular values are arranged on the main diagonal of the matrix Σ in a descending order. Subscript indices were introduced into equation (2.15) in order to avoid confusion with system inputs and outputs notation. $\mathbf{U}_y$ is an orthogonal $l \times m$ matrix of output singular vectors $\mathbf{u}_{y_i}$. $\mathbf{V}_u$ is an orthogonal $m \times m$ matrix of input singular vectors $\mathbf{v}_{u_i}$. The Singular Value Decomposition matrices for a general 2×2 system are

$$\mathbf{G} = \underbrace{\begin{bmatrix} \cos\phi_1 & -\sin\phi_1 \\ \sin\phi_1 & \cos\phi_1 \end{bmatrix}}_{\mathbf{U}_y} \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} \cos\phi_2 & \pm\sin\phi_2 \\ -\sin\phi_2 & \pm\cos\phi_2 \end{bmatrix}^T}_{\mathbf{V}_u^T}, \tag{2.16}$$

where the angles ϕ_1, ϕ_2 depend on the given system gain matrix $\mathbf{G}$. From equation (2.16) it can be seen, that the matrices $\mathbf{U}_y$ and $\mathbf{V}_u^T$ involve rotations and their columns are orthogonal. The matrix Σ involves system singular values. The columns of matrix $\mathbf{V}_u$ represent the input directions, so an input signal is rotated by the transpose of this matrix. The rotated vectors are then amplified by the matrix Σ, and then rotated again at the system output by the matrix $\mathbf{U}_y$. As the column vectors of $\mathbf{V}_u$ (and $\mathbf{U}_y$) representing the input (and output) are orthogonal and of unit length, a singular value σ_i directly gives the gain of matrix $\mathbf{G}$ in their direction. If we denote $\mathbf{v}_{ui}$ a column vector of the $\mathbf{V}_u$ matrix, we can write

$$\sigma_i(\mathbf{G}) = \|\mathbf{G}\mathbf{v}_{ui}\|_2 = \frac{\|\mathbf{G}\mathbf{v}_{ui}\|_2}{\|\mathbf{v}_{ui}\|_2} . \tag{2.17}$$

Comparing to system eigenvalue decomposition, the advantage of SVD is that it delivers more accurate information on the gains of the plant. Another important property of SVD is the fact that it can be applied to non-square system matrices [1]. When working with non-square systems with higher number of outputs than inputs, the benefit of using SVD is the fact, that it can be easily determined in which direction is the plant difficult to control. On the other hand, when a system has more inputs than outputs, the directions where the input has no effect on the output may be determined. SVD is also often used in order to evaluate the condition number $\gamma(\mathbf{G})$ of a system, as the highest and smallest system singular values, which are used for condition number calculation, are easily found on the Σ matrix diagonal.

2.2.2 Relative Gain Array

For a non-singular system with square system gain matrix **G** a complex square matrix called Relative Gain Array (RGA) is defined as

$$\mathrm{RGA}(\mathbf{G}) = \Lambda(\mathbf{G}) \triangleq \mathbf{G} \times (\mathbf{G}^{-1})^T \ , \tag{2.18}$$

where symbol $'\times'$ represents an element-by-element multiplication. The Relative Gain Array is calculated as a function of frequency. Originally, it was developed by Edgar Bristol as a measure of system interactions in 1966 [5]. A multivariable plant **G** has input-output pairs u_j and y_i, where we assume u_j controls y_i. The idea of RGA is to split the analysis into two sub-problems. At first, only the appropriate input-output pair is closed into a loop, all the other loops in the system are opened $(u_k = 0,\ \forall k \neq i)$. The second case represents the situation when all the loops in the system are closed with perfect control $(y_k = 0,\ \forall k \neq i)$. The system gain is calculated for both of these cases, denoted g_{ij} for the first case with only one closed loop in the system, and $\hat{g}_{ij}$ for the case with all system loops closed and a perfect control. A relative gain can be now evaluated as the ratio of these two gains

$$\lambda_{ij} \triangleq \frac{g_{ij}}{\hat{g}_{ij}} \ . \tag{2.19}$$

This relative gain should be calculated for all the possible system input-output pairs and organized into a square matrix, denoted $\mathrm{RGA}(\mathbf{G})$.

The Relative gain array is a useful tool when deciding on pairing of inputs and outputs. A decision on which input to use for a control of particular output may be done according to the size of RGA elements. The pairings of inputs and outputs in the system should be chosen so that the rearranged system with the selected pairings along the diagonal has an RGA matrix close to identity at frequencies around the closed-loop bandwidth. An emphasis should be taken on the possibility of a change in a sign of the steady state gain depending on the control of other outputs. Pairing of negative steady-state elements should be avoided if possible. As the RGA matrix depends on frequency, it sometimes shows out to be useful to evaluate the RGA elements over the whole frequency range. This analysis may show that different input-output pairings should be chosen for different frequency regions, but for most systems this is not the case of usual use of RGA.

Another useful concept has been developed from Rosenbrocs´ Relative Gain Array. The RGA shows to be not only frequency dependent, but the phase of RGA elements should be considered as well. This problem overcomes a concept of RGA number. The definition of this measure is

$$\text{RGAnumber} \triangleq \|\Lambda(\mathbf{G}) - \mathbf{I}\|_{sum} \ , \tag{2.20}$$

where the sum norm, defined as $\|\mathbf{A}\|_{sum} = \Sigma_{i,j} \left|a_{ij}\right|$, is used. The RGA number is evaluated for a selected input-output pairing. A matrix with ones on the position of interconnected inputs and outputs is substracted from the system RGA matrix. For an off-diagonal pairing of a system with 2x2 gain matrix $\mathbf{G}$ the RGA number would be calculated as $\text{RGAnumber} = \Lambda(\mathbf{G}) - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. A drawback of using RGA number for large systems is the fact, that it has to be calculated for each possible alternative pairing of inputs and outputs. The preferred pairing is indicated by the RGA number close to identity.

At the end of this section, the most important properties of RGA should be concluded, as it is one of the most popular system interaction measures [1]. An important advantage of RGA is the independency of this measure on input and output scaling. It is a straight measure of ill-conditioning of a plant, as plant with large RGA elements at frequencies important for control indicate that the plant is fundamentally difficult to control due to strong interactions. Plants with large RGA elements are fundamentally difficult to control because of sensitivity to input uncertainty. It is possible to indicate a right half-plane zero of the system using RGA. If the sign of an element of RGA changes when moving from $s = 0$ to $s = \infty$, the system has a right half-plane zero. On the other hand side, the biggest disadvantage of RGA is that it is frequency-dependent.

2.2.3 Condition number

Together with the Relative Gain Array, the condition number is one of the basic tools to quantify the system directionality. As mentioned in section 2.2, the condition number is the ratio between the largest singular value $\overline{\sigma}(\mathbf{G}(j\omega))$ and smallest singular value $\underline{\sigma}(\mathbf{G}(j\omega))$ of a system. System condition $\gamma(\mathbf{G}(j\omega))$ is calculated using the equation (2.13). Large condition number indicates that there is a large difference in the system gain depending on input and output directions.

Scaling of the inputs and outputs is very important when interpreting the condition number. The gain matrix $\mathbf{G}$ should be scaled on physical grounds, which is done by dividing each input and output by its largest expected or desired value.

The condition number might be used as an input - output controllability measure. Large condition number indicates sensitivity to uncertainty. This property can not be used in general, but the reverse holds. If the condition number is small, the multivariable effects of uncertainties are not likely to be serious. A large condition number $\gamma(\mathbf{G}(j\omega))$ may be caused by a small value of $\underline{\sigma}(\mathbf{G}(j\omega))$, which is generally undesirable. A benefit of using condition number is its property that it indicates sensitivity to unstructured input uncertainty. Although the condition number is a measure of ill - conditioning of a system, there is no exact value of this indicator that would be considered as a threshold value between well-conditioned and ill-conditioned systems. In most cases the systems with $\gamma(\mathbf{G}(j\omega)) \gg 10$ are considered to be ill-conditioned [1].

At this point, it is interesting to present also a different type of condition number that will be used in this thesis. The above explained system directionality influences the ability of a controller to compensate disturbances. Due to the system directionality some disturbances are more difficult to reject than the others. As output disturbances will be considered for simulations in section 3.2.4, a disturbance condition number for this type of disturbance will be presented. A disturbance vector $\mathbf{d}$ which has a direction close to a system output direction $\bar{\mathbf{u}}_y$, corresponding to the system largest singular value $\overline{\sigma}(\mathbf{G}(j\omega))$ is expected to be easy to reject. On the other hand, a disturbance in a direction close to the weak output direction $\underline{\mathbf{u}_y}$ is expected to be difficult to reject. A measure called disturbance condition number $\gamma_d(\mathbf{G})$ gives a more precise measure of how the disturbance is aligned with the plant output direction [3]. The disturbance condition number is calculated as

$$\gamma_d(\mathbf{G}) = \frac{\left\|\mathbf{G}^{-1}\mathbf{d}\right\|_2}{\|\mathbf{d}\|_2}\overline{\sigma}(\mathbf{G}) \, , \tag{2.21}$$

where $\mathbf{d}$ is the disturbance vector and $\bar{\sigma}(G)$ is the largest singular value of a system. $\gamma_d(G)$ ranges in magnitude between 1 and the system condition number $\gamma(\mathbf{G})$. A value close to $\gamma(\mathbf{G})$ indicates that the disturbance is in undesired direction, close to the direction corresponding to the smallest plant singular value $\underline{\sigma}(\mathbf{G})$. A disturbance condition number close to 1 indicates closeness to the direction of largest system singular value $\overline{\sigma}(\mathbf{G})$, which is easier to reject.

2.2.4 Gramian based interaction measures

All the previous measures of system directionality are mainly used in decentralized control. These methods were mostly developed in the era between 1960's and the end of 20^{th} century [5, 4]. In the first decade of 21^{st} century, a new interaction measures were used in order to quantify the system interactions and to enable the engineers to use alternative controller structures. Gramian based methods, which are a modern approach to MIMO system interactions study, are based upon a dynamic model of a process. The advantage of these interaction measures is that they quantify interactions as a function of chosen channel bandwidths, and they give criteria for input-output pairing. This allows control engineers to assess alternative controller architectures [10].

Participation Matrix (PM) and Hankel Interaction Index Array (HIIA) are the two basic gramian based methods. There are many modifications of these methods, but these two methods are a good example of the principle of Gramian based interaction measures. As the name of these methods suggests, they are based on Gramians, which are matrices that describe certain controllability and observability properties of a stable system. It is possible to find a Gramian of both continuous and discrete system. There are two types of Gramian - Controllability Gramian and Observability Gramian. For a system with the same number of system inputs and outputs ($m = l$) with a state space description with state matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, the controllability Gramian $\mathbf{P}_{ctrl}$ and observability Gramian $\mathbf{Q}_{obs}$ satisfy the Lyapunov equations

$$\begin{aligned}\mathbf{A}\mathbf{P}_{ctrl} + \mathbf{P}_{ctrl}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0 \ , \\ \mathbf{A}^T\mathbf{Q}_{obs} + \mathbf{Q}_{obs}\mathbf{A} + \mathbf{C}^T\mathbf{C} = 0 \ .\end{aligned} \tag{2.22}$$

Gramians quantify how hard it is to control and observe the state of the system. The ranks of $\mathbf{P}_{ctrl}$ and $\mathbf{Q}_{obs}$ are the dimensions of the controllable and observable system subspace. As there are many state space descriptions for a system, gramians are dependent on the particular state space realization. To eliminate this dependence, a product of the controllability and observability Gramians $\mathbf{P}_{ctrl}\mathbf{Q}_{obs}$ is evaluated. The eigenvalues $\lambda_i(i = 1, 2, ...n)$ of this product are called Hankel Singular Values (HSV) of the system. These non-negative values are independent of particular system realization. The product $\mathbf{P}_{ctrl}\mathbf{Q}_{obs}$ can be calculated for all the input-output combinations of the system separately: $\mathbf{P}_{ctrl}\mathbf{Q}_{obs} = \sum_{i,j=0}^{m} \mathbf{P}_{ctrl_i}\mathbf{Q}_{obs_j}$. This decomposition may be explained as a combination of all the SISO systems which are combined in the MIMO original system. A trace of $\mathbf{P}_{ctrl_i}\mathbf{Q}_{obs_j}$ is equal to the sum of the Hankel Singular Values of the elementary system. Moreover, this product is a convenient basis to measure the interaction and the ability of different controller structures to control and observe the system state. This measure can be organized into a matrix $\phi = [\phi_{ij}] \in \mathbb{R}^{\mathrm{m}\times\mathrm{m}}$, which is called Participation Matrix, defined by

$$\phi_{ij} = \frac{\mathrm{trace}[\mathbf{P}_{ctrl_i}\mathbf{Q}_{obs_j}]}{\mathrm{trace}[\mathbf{P}_{ctrl}\mathbf{Q}_{obs}]} \leq 1 \ , \tag{2.23}$$

where the denominator normalizes the trace measure. The sum of all elements ϕ_{ij} is equal to one, thus the complexity of a controller should be traded off against closeness of the corresponding ϕ_{ij} elements sum to unity. In addition to this criterion, a controller with minimum complexity and even a decentralized controller may be the result of this method.

The Hankel Interactive Index Array uses an important property of a system, which is the fact that the controllability and observability Gramians for the full system are a sum of Gramians for all the subsystems. If a Hankel norm is calculated for each subsystem and arranged in a matrix $\tilde{\Sigma}_H$ given by

$$\left[\tilde{\sum}_H\right]_{ij} = \|\mathbf{G}_{ij}\|_H \ , \tag{2.24}$$

the Hankel norm of a system is calculated as

$$\|\mathbf{G}\|_H = \sqrt{\lambda_{max}(\mathbf{P}_{ctrl}\mathbf{Q}_{obs})} = \sigma_1^H \ , \tag{2.25}$$

where σ_1^H is the maximum Hankel Singular Value. The $\tilde{\Sigma}_H$ matrix from equation (2.24) can be used as an interaction measure. A normalized version is the Hankel Interaction Index Array

$$\left[\sum{}_H\right]_{ij} = \frac{\|\mathbf{G}_{ij}\|_H}{\sum_{kl}\|\mathbf{G}_{kl}\|_H} \ . \tag{2.26}$$

With the normalization, the sum of elements in Σ_H is one. The larger the element, the larger the impact of the corresponding input signal on the specific output signal. Expected performance for different controller structures can be compared by summing the corresponding elements in Σ_H. The aim is to find the simplest control structure that gives a sum as close to 1 as possible [11]. The main difference between HIIA and PM is the fact that HIIA is given only by the largest

Hankel singular value, whereas PM considers all Hankel singular values of a system, therefore it uses more information on the system.

There are some other methods that may be used as MIMO system interaction measures. Most of these methods are based on the methods described in sections 2.2.2 and 2.2.4. Some of these methods are described in the background research presented in the Appendix A of this thesis.

2.2.5 Comparison of interaction measures

The important properties of different system interaction measures were described in the previous paragraphs. It is useful to make a comparison of these methods at the end of this section. In this thesis an MPC for ill-conditioned plants will be designed, so the directionality measures of a system are in our main interest. The Singular Value Decomposition is definitely a useful tool when it comes to a system directionality analysis, as it provides information on both the system conditioning and the directions of the system singular values. RGA, and the Gramian based methods are a different type of system analysis tools. These methods are mainly used to find an appropriate input-output couplings for a diagonalized MIMO controller. Although this is not the aim of this thesis, these methods also show the interactions within a system, which is a useful information on a MIMO systems.

Comparing the properties of the interaction measures, the widely used RGA shows to be a good method for decentralized control structure input-output pairing. As it is one of the first and well-known methods of this type, there has been many modifications done to overcome some of its disadvantages. The main disadvantage is the fact, that it is evaluated for a particular frequency. Moreover it is not able to cope with non-minimum phase structures and is insensitive to delays. The modern Gramian based measures are based upon a dynamic model of a process [22]. These methods are frequency independent. The PM is able to detect time delays, but it may be affected by time delays as shown in [11]. The biggest advantage of PM over the other methods is the fact that in addition to choosing input-output pairs for diagonalized controllers, it is possible to use this method for choosing input-output combinations for multivariable controllers with non-diagonal input-output interconnections. The drawback of HIAA is its scaling dependency.

Use of more than one system directionality and interaction analysis method at a time is a possibility of how to overcome some disadvantages of the individual methods. It is also a useful approach when the result of one method is confirmed by results of the other methods.

2.3 Control of multivariable plants

Only a brief review of control methods that are widely used in praxis will be given in this section. Emphasis will be given on Model Predictive Control, which will be used for ill-conditoned plants control in the next chapter of this thesis.

2.3.1 Decentralized control of multivariable systems

The concept of decentralized (diagonal) control is based on a fact, that in many cases a MIMO system gain matrix $\mathbf{G}(s)$ is close to diagonal, so the system can be controlled as a collection of independent subsystems. In such case a diagonal controller $\mathbf{K}(s)$ may be used. If the off-diagonal elements of $\mathbf{G}(s)$ are large, then the performance with decentralized diagonal control may be poor, because no attempt is made to compensate the interactions.

To use diagonal control, a plant can be modified by a pre-compensator. This type of control is called a two-step control design. In the first step of this approach a compensator $\mathbf{W}_1$ is designed. This compensator deals with the interactions in $\mathbf{G}(s)$. In the second step a controller $\mathbf{K}(s)$ is designed for the compensated system $\mathbf{G}_s(s) = \mathbf{G}(s)\mathbf{W}_1(s)$. The pre-compensator counteracts the interactions in the system, so a controller $\mathbf{K}_s$ can be designed for the system $\mathbf{G}_S(s)$ using the SISO systems controller design techniques. The fact, that the SISO techniques can be used for the controller design is a reason of popularity of decentralized control in industry [10]. The overall controller is of the form

$$\mathbf{K}_s = \mathbf{W}_1(s)\mathbf{K}_s(s) \tag{2.27}$$

As mentioned in the previous paragraph, to use diagonal control, decoupling of the system has to be done. There are different decoupling techniques, which are used, depending on a particular system. Dynamic decoupling results in $\mathbf{G}_s$ diagonal at all frequencies. This type of decoupling often results in an inverse based controller, which may be a problem when looking for the system realization. Steady state decoupling is often achievable using a constant pre-compensator $\mathbf{W}_1 = \mathbf{G}(0)^{-1}$. If the system matrix is non-square, but full rank, pseudo-inverse may be used to get $\mathbf{W}_1$. Last of the decoupling techniques is an approximate decoupling at frequency ω_0. $\mathbf{G}_s(j\omega)$ as diagonal as possible is usually obtained by choosing a constant pre-compensator $\mathbf{W}_1 = \mathbf{G}_0^{-1}$, where $\mathbf{G}_0$ is a real approximation of $\mathbf{G}(j\omega)$. Decoupling control has some crucial limitations. Decoupling is very sensitive to modelling errors and uncertainties. As already mentioned, a pre-compensator is often an inverse based controller, which may not be desirable for disturbance rejection. An alternative approach to classical decoupling methods is partial decoupling, where $\mathbf{G}_s(s)$ is upper or lower triangular.

The presented pre-compensator approach may be extended by introducing a post-compensator $\mathbf{W}_2(s)$. The resulting controller is of the form $\mathbf{K} = \mathbf{W}_1\mathbf{K}_s\mathbf{W}_2$. A special case of this type of controller is the SVD controller. The pre-compensator and post-compensator are derived from the Singular Value Decomposition of a system. Using the equation (2.16) for the SVD, we can make a real approximation of $\mathbf{G}(j\omega_0)$ at a given frequency as $\mathbf{G}_0 = \mathbf{U}_{y_0}\Sigma\mathbf{V}_{u_0}$. From this approximation we can get the pre-compensator as $\mathbf{W}_1 = \mathbf{V}_{u_0}$ and post-compensator as $\mathbf{W}_2 = \mathbf{U}_{y_0}$.

2.3.2 Multivariable controller synthesis - H_∞ control

An alternative to the two-step controller design described in the previous paragraph is to directly design a multivariable controller $\mathbf{K}(s)$ based on minimizing some objective function (norm). This approach is often called a controller synthesis, rather than controller design. One of the widely used controller synthesis is the mixed sensitivity $(\mathbf{S}/\mathbf{KS})$ H_∞ synthesis.

In the mixed sensitivity approach the system sensitivity function $\mathbf{S} = (\mathbf{I}+\mathbf{GK})^{-1}$ is shaped along with one or more other closed-loop transfer functions such as $\mathbf{KS}$, which is a transfer function between disturbance $\mathbf{d}_1$ (displayed in Figure 2.1) and the control signals, or with the complementary sensitivity function $\mathbf{T} = \mathbf{I} - \mathbf{S}$. Limiting $\mathbf{KS}$ is important, as it is a mechanism for limiting the size and bandwidth of the controller. Common aim is to reject disturbance $\mathbf{d}_1$, which is typically a low-frequency signal. This type of disturbance is rejected if the maximum singular value of $\mathbf{S}$ is made small over the same low frequencies where disturbance $\mathbf{d}$ occurs. To achieve this, a scalar low-pass filter $w_1(s)$ with a bandwidth equal to that of disturbance has to be selected. Then a stabilizing controller, that minimizes $\|w_1\mathbf{S}\|_\infty$ has to be found. This type of controller focuses on just one type of closed-loop transfer function and for plants with right half-plane zero has infinite gain. Therefore it is useful to minimize $\left\| \begin{matrix} w_1\mathbf{S} \\ w_2\mathbf{KS} \end{matrix} \right\|_\infty$ where w_2 is a scalar high-pass filter with a crossover frequency approximately equal to the frequency of desired closed-loop bandwidth.

The objective in the S/KS problem is to minimize the H_∞ norm of

$$\mathbf{N} = \begin{bmatrix} \mathbf{W}_P\mathbf{S} \\ \mathbf{W}_U\mathbf{KS} \end{bmatrix} \tag{2.28}$$

where $\mathbf{S}$ is the same system output sensitivity as described by equation (2.1). The weights $\mathbf{W}_P$ and $\mathbf{W}_U$ may be selected according to guidelines in Chapter 3 of [1]. A common choice of the weight $\mathbf{W}_P$ is $\mathbf{W}_P = \mathrm{diag}(w_{Pi})$ with

$$w_{Pi} = \frac{s/\mathbf{M}_i + \omega_{Bi}^\star}{s + \omega_{Bi}^\star \mathbf{A}_i} \tag{2.29}$$

where $\omega_{Bi}^\star$ is the desired closed - loop bandwith, which may be different for each output. The variables $\mathbf{M}_i$ and $\mathbf{A}_i$ are selected according to the desired controller performance, for example to approximate integral action, the value $\mathbf{A}_i \ll 1$ should be chosen. For a scaled system, reasonable choice of initial input weight $\mathbf{W}_u$ is $\mathbf{W}_u = \mathbf{I}$.

2.3.3 LQ control

Linear Quadratic Regulator (LQR) is a popular regulator suitable for control of linear systems. The regulator design procedure is based on Bellman' s principle of optimality. It uses the fact, that no matter what the actual state of the system is, the following control steps must be optimal. Bellman' s principle of optimality is used together with a principle of invariant embedding. This principle describes the fact, that a solution of a set of problems includes solutions of all the problems within the solved set. Taking advantage of these two principles an optimality criterion in quadratic form is solved. For a system with state-space description

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \ , \end{aligned} \tag{2.30}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ is the input vector, $\mathbf{y}(t) \in \mathbb{R}^p$ is the output vector, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are real $n \times n$, $n \times m$, $p \times n$ and $p \times m$ matrices, the whole procedure results in a control law of the form

$$\mathbf{u}^{\star}(t) = -\mathbf{K}(t)\mathbf{x}(t)\ , \tag{2.31}$$

where $\mathbf{K}(t)$ is the Kalman gain of a continuous LQ regulator. To derive this optimal system input it is necessary to build a cost function, which describes requirements on the system behaviour. The cost function consists of a final state term and a term with all the states that lead towards the terminal state, starting at the initial point. For the control interval $k_0 \in \langle 0;\, N\rangle$ the optimality criterion is

$$J = \frac{1}{2}\mathbf{x}^T(t_1)\mathbf{S}\mathbf{x}(t_1) + \frac{1}{2}\sum_{t=t_0}^{t_1-1}\{\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + \mathbf{u}^T(t)\mathbf{R}\mathbf{u}(t)\}\ , \tag{2.32}$$

where $\mathbf{S} \in \mathbb{R}^{n\times n}$ and $\mathbf{Q} \in \mathbb{R}^{\mathbf{n}\times\mathbf{n}}$ are positive semi-definite constant matrices and $\mathbf{R} \in \mathbb{R}^{m\times m}$ is positive definitive matrix. These matrices are used for controller tuning. As the cost function is optimized, the weighting matrices penalize the individual terms of the cost function. Matrix $\mathbf{S}$ penalizes the final state cost, matrix $\mathbf{Q}$ the system states and matrix $\mathbf{R}$ is used for input value penalization. For an increasing optimality horizon $(t_1 \rightarrow \infty)$ is the Kalman gain approaching to constant matrix $\mathbf{K}$ and also matrix $\mathbf{P}(t)$ given by Ricatti equation is approaching to constant matrix $\mathbf{P}$. This matrix is the solution of Algebraic Ricatti Equation

$$\mathbf{P} = \mathbf{A}^T\mathbf{P}\mathbf{A} - \mathbf{A}^T\mathbf{P}\mathbf{B}[\mathbf{R} + \mathbf{B}^T\mathbf{P}\mathbf{B}]^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A} + \mathbf{Q}\ . \tag{2.33}$$

Linear optimal Kalman gain then equals

$$\mathbf{K} = [\mathbf{R} + \mathbf{B}^T\mathbf{P}\mathbf{B}]^{-1}\mathbf{B}^T\mathbf{P}\mathbf{A}\ . \tag{2.34}$$

Linear Quadratic Regulator with the above calculated Kalman gain exists only if the controlled system is stabilizable and detectable with the output matrix $\mathbf{C}_Q$, where $\mathbf{C}_Q^T\mathbf{C}_Q = \mathbf{Q}$ [2].
A combination of LQ controller and a Kalman filter is widely used modification of basic LQ controller, known as LQG controller.

2.3.4 Model Predictive Control

Model predictive control is a modern approach to systems control, which in contrast to many of the other control strategies has been used in practise before it has been precisely mathematically described. The first formulation of a moving horizon control comes from late 1960's [20]. As it proved as a useful control strategy, it is now a standard advanced control technology widely used in industry. A brief introduction to Model Predictive Control will be given in this chapter, based on [2, 18, 19].

The popularity of MPC is partly caused by the fact, that it enables optimal control of constrained multivariable dynamical systems. As the name suggests, MPC is a control approach based on a system model. Goals for the system control are formulated in terms of a cost function. The cost function is usually in an additive form, where the individual terms express various control demands. Having the system model and the cost function, the sequence of input vectors $\mathbf{u}^{\star N}_{k} = \left[\mathbf{u}^{\star T}_{k}, \mathbf{u}^{\star T}_{k+1}, \dots \mathbf{u}^{\star T}_{k+N}\right]^{T}$ for a horizon length N is achieved by the cost function optimization. The significant property of Model Predictive Control, which differentiates it from other control strategies, is the fact, that there may be constrains introduced in the cost function. In contrast with above mentioned multivariable control methods, MPC does not result in the state feedback control law, but directly in an optimal future trajectory of the system input $\mathbf{u}^{\star N}_{k}$. The sequence of input vectors is calculated for a control horizon, which is chosen by the control system designer. The whole calculated sequence of the system input may be applied to the system. This approach is an equivalent to the open loop control, as no disturbances and uncertainties of the system model and measurements are taken into account. Although more computationally demanding, an equivalent to the closed loop control is the Receding Horizon Control, where the optimization problem is computed at each sampling period, after having new system measurement or estimates. Only the first control action from the sequence of optimized input vectors $\mathbf{u}^{\star N}_{k}$ is applied.

The fundamental stages in Model Predictive control design will be described in the following paragraphs. The system model and cost function, which is usually composed of additive terms, representing control system demands and constraints, are both created at the design stage of MPC. During the control procedure itself, the optimization is performed at each step in the receding horizon control. Although these are three different parts of MPC design, they can not be designed separately as they are closely related. For example the cost function has to be designed so that it is possible to optimize it in the later stage.

System model

System model and its quality play an important role in Model Predictive Control. Without a good system model, it is very difficult to control a system with an MPC controller, even if the remaining compounds of the regulator are perfectly designed. When considering linear MPC, there is no limitation on which kind of system representation should be chosen. The commonly used models are for example ARX model, impulse response model, step response based model or a state space model. Most common is the state space model, therefore, it will be considered in this text. The state space model has been chosen not only for its popularity within control community, but also for the property, that all the other mentioned system representations may be converted into a state space model. A general state space model of a discrete system is described by a set of state space equations

$$\begin{aligned} \mathbf{x}_{k+1} &= \mathbf{A}\mathbf{x}_k + \mathbf{B}\mathbf{u}_k \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \mathbf{D}\mathbf{u}_k \ , \end{aligned} \tag{2.35}$$

where $\mathbf{x}_k \in \mathbb{R}^n$ is the state vector, $\mathbf{u}_k \in \mathbb{R}^m$ is the input vector, $\mathbf{y}_k \in \mathbb{R}^p$ is the output vector, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are real $n \times n$, $n \times m$, $p \times n$ and $p \times m$ matrices. The state predictions for this system are given by

$$\mathbf{x}_{k+1}^{k+N+1} = \mathbf{P}\mathbf{x}_k + \mathbf{H}\mathbf{u}_k^{k+N} \ , \tag{2.36}$$

where the vectors $\mathbf{x}_{k+1}^{k+N+1}$, $\mathbf{u}_k^{k+N}$ and matrices $\mathbf{P}$ and $\mathbf{H}$ are

$$\begin{aligned} \mathbf{x}_{k+1}^{k+N+1} &= \left[\mathbf{x}_{k+1}^T,\ \mathbf{x}_{k+2}^T, \ldots,\ \mathbf{x}_{k+N+1}^T\right]^T \ , \\ \mathbf{u}_k^{k+N} &= \left[\mathbf{u}_k^T,\ \mathbf{u}_{k+1}^T, \ldots,\ \mathbf{u}_{k+N}^T\right] \end{aligned} \tag{2.37}$$

$$\mathbf{P} = \begin{bmatrix} \mathbf{A} \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^N \end{bmatrix}, \ \mathbf{H} = \begin{bmatrix} \mathbf{B} & & & \\ \mathbf{A}\mathbf{B} & \mathbf{B} & & \\ \vdots & & \ddots & \\ \mathbf{A}^{N-1}\mathbf{B} & \mathbf{A}^{N-2}\mathbf{B} & \cdots & \mathbf{B} \end{bmatrix} . \tag{2.38}$$

The prediction trajectories of the Model Predictive Controller will be in the form

$$\mathbf{y}_k^{k+N} = \overline{\mathbf{P}}\mathbf{x}_k + \overline{\mathbf{H}}\mathbf{u}_k^{k+N} \tag{2.39}$$

where $\mathbf{x}_k$ is the initial state for a prediction horizon N and $\mathbf{y}_k^{k+N} = \left[\mathbf{y}_k^T,\, \mathbf{y}_{k+1}^T, \ldots,\, \mathbf{y}_{k+N}^T\right]^T$ is the vector of outputs on the prediction horizon. The matrices $\overline{\mathbf{P}}$ and $\overline{\mathbf{H}}$ are

$$\overline{\mathbf{P}} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{N-1} \end{bmatrix},\ \overline{\mathbf{H}} = \begin{bmatrix} \mathbf{D} & & & \\ \mathbf{CB} & \mathbf{D} & & \\ \vdots & & \ddots & \\ \mathbf{CA}^{N-2}\mathbf{B} & \cdots & \mathbf{CB} & \mathbf{D} \end{bmatrix}. \tag{2.40}$$

Cost function

In terms of achieving desired behaviour, the cost function is a key component of Model Predictive Controller. The cost function is usually of an additive form, composed of individual terms that define different requirements and soft or hard constraints on the system behaviour. If the cost function is well designed, the control goal is achieved by the cost function optimization during the regulator run. The individual criteria within the cost function are multiplied by weighting coefficients, which assign relative importance to the terms.

There might be different criteria and requirements, but often reference tracking is the basic requirement. To ensure reference tracking, the corresponding cost function has to penalize tracking error over the chosen prediction horizon. The second basic term is usually the term that specifies actuator behaviour. The standard basic cost function, taking into account these two criteria is expressed as

$$J\left(\mathbf{u}_k^{k+N} | \mathbf{x}_k, \mathbf{r}_k\right) = \sum_{i=k}^{k+N} \left\|(\mathbf{y}_i - \mathbf{r}_i)^T \mathbf{Q} (\mathbf{y}_i - \mathbf{r}_i)\right\|_p + \sum_{j=k}^{N_u-1} \left\|\mathbf{u}_j^T \mathbf{R} \mathbf{u}_j\right\|_p, \tag{2.41}$$

where $\mathbf{r}_k$ is output reference vector at time k and matrices $\mathbf{Q} \in \mathbb{R}^{p\times p}$ and $\mathbf{R} \in \mathbb{R}^{m\times m}$ are the weighting matrices which represent the relative importance of criteria. Matrix $\mathbf{Q}$ penalizes the tracking error, and matrix $\mathbf{R}$ penalizes the input value. $\mathbf{u}_k^{k+N}$ is the system input vector and $\mathbf{x}_k$ is the initial state of the system. The l_p norm of a vector x of length n is defined as

$$\|\mathbf{x}\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}\,. \tag{2.42}$$

The first sum in equation (2.41) represents the term that penalizes the tracking error with relative weight $\mathbf{Q}$ and the second sum penalizes the actuator position with relative weight $\mathbf{R}$.

A modification of cost function described by equation (2.41), which penalizes the input value increment instead of its position is often used in practical applications. This type of MPC is sometimes called a minimum movement controller [18]. The second sum of the cost function is changed, so the cost function has the following form:

$$J\left(\mathbf{u}_k^{k+N}|\mathbf{x}_k,\mathbf{r}_k\right)=\sum_{i=k}^{k+N}\left\|(\mathbf{y}_i-\mathbf{r}_i)^T\mathbf{Q}(\mathbf{y}_i-\mathbf{r}_i)\right\|_p+\sum_{j=k}^{N_u-1}\left\|\Delta\mathbf{u}_j^T\mathbf{R}\Delta\mathbf{u}_j\right\|_p\ , \tag{2.43}$$

where $\Delta\mathbf{u}_j=\mathbf{u}_j-\mathbf{u}_{j-1}$.

The cost function criteria have to be added carefully, as the cost function is optimized in the following step. The optimization method depends on the chosen l_p norm. The conventional norms are $l_{1,}$ l_2 and l_∞. The l_2 norm is widely used and ensures good performance of the control loop, therefore we will consider it in this text. Utilization of this norm leads to Quadratic Programming, whereas utilization of l_∞ norm leads to Linear Programming.

Constraints

The possibility of introducing constraints in the controller differentiates MPC from other conventional control approaches. All realistic processes have some constraints. In some cases these could be actuator position constraints

$$\mathbf{u}_{min_k}\leq\mathbf{u}_k\leq\mathbf{u}_{max_k}\ , \tag{2.44}$$

rate of input change constraints

$$\triangle\mathbf{u}_{min_k}\leq\triangle\mathbf{u}_k\leq\triangle\mathbf{u}_{max_k}\ , \tag{2.45}$$

output constraints

$$\mathbf{y}_{min_k}\leq\mathbf{y}_k\leq\mathbf{y}_{max_k}\ , \tag{2.46}$$

state constraint

$$\mathbf{x}_{min_k}\leq\mathbf{x}_k\leq\mathbf{x}_{max_k}\ , \tag{2.47}$$

or some other constraints. As described in the previous paragraph, constraints may be introduced into the optimization problem as modifications or additional terms of the cost function. There are generally two types of constraints, namely the hard constraints and the soft constraints. The hard constraints are involved because of physical limitations of real processes, as for example the extreme positions of valves. These constraints must not be violated during the system run. On the other hand, the soft constraints represent the constraints that may be violated but with some penalty. This penalty evaluates how much the variable can differentiate from the limiting value. The soft constraint terms in the cost function may represent for example a soft limit for the speed or quality of production. Both may exceed their maximum value, but have to be penalized for it so that this maximum value violation is limited.

Soft constraints are very important for all practical implementations, because they ensure feasibility of MPC optimization problem when there are disturbances acting on the controlled process. This is a common phenomenon in practical applications. The soft constraints are for example formulated by introducing a loose optimization variable or vector. A soft upper limit for a system output may be formulated as

$$y_{k_i} \leq y_{max} + \varepsilon \ , \tag{2.48}$$

where the variable ε is a scalar variable. When using the l_2 norm, there is a term $\|\varepsilon\|_2^2$ added to the cost function to include the soft constraint. As mentioned earlier, the soft constraints may be violated, which happens mainly during the transients and during disturbance rejection. The weighting factor of the soft constraints must be high enough to ensure an acceptable variance only. The individual soft constraints of the system may be weighted separately, or all by the same slack factor.

Optimization

The control law known from linear state feedback theory as $\mathbf{u}_k = \mathbf{K}\mathbf{x}_k + \mathbf{g}$ is not a result of MPC. The optimal system input is calculated for the whole horizon, based on the cost function optimization, which ensures taking into account all the necessary system constraints. The control problem of MPC can therefore be formulated as an optimization problem

$$\mathbf{u}_k^{\star k+N} = \arg\min_{\vec{u}} J\left(\mathbf{u}_k^{k+N} | \mathbf{x}_k, \mathbf{r}_k\right) \ , \tag{2.49}$$

with input constraints

$$\mathbf{u}_{min_{k+i}} \leq \mathbf{u}_{k+i} \leq \mathbf{u}_{max_{k+i}} \ , \tag{2.50}$$

or in case of minimum movement controller

$$\triangle\mathbf{u}_{min_{k+i}} \leq \triangle\mathbf{u}_{k+i}\triangle\mathbf{u}_{max_{k+i}} \ , \tag{2.51}$$

output constraints

$$\mathbf{y}_{min_{k+i}} \leq \mathbf{y}_{k+i} \leq \mathbf{y}_{max_{k+i}} \ , \tag{2.52}$$

system state constraints

$$\mathbf{x}_{min_{k+i}} \leq \mathbf{x}_{k+i} \leq \mathbf{x}_{max_{k+i}} \ , \tag{2.53}$$

and system model equations

$$\begin{aligned} \mathbf{x}_{i+1} &= \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i \ , \\ \mathbf{y}_i &= \mathbf{C}\mathbf{x}_i + \mathbf{D}\mathbf{u}_i \ , \end{aligned} \tag{2.54}$$

where the output constraints, and state constraints are usually softened.

If the l_2 norm is used in the cost function, the MPC problem for a linear system with constraints can be transformed to a mathematical programming problem. Using equation 2.39, the optimization problem is

$$\mathbf{u}_k^{\star k+N} = \arg\min_{\mathbf{u}_k^{k+N}} \frac{1}{2}\mathbf{u}_k^{k+N T}\mathbf{G}\mathbf{u}_k^{k+N} + \mathbf{u}_k^{k+N T}\mathbf{F}\mathbf{p} \ , \tag{2.55}$$

where

$$\begin{aligned} \mathbf{F} &= \bar{\mathbf{H}}^T\mathbf{Q}\bar{\mathbf{P}} \ , \\ \mathbf{G} &= \bar{\mathbf{H}}^T\mathbf{Q}\bar{\mathbf{H}} + \mathbf{R} \ , \end{aligned} \tag{2.56}$$

which is a quadratic programming problem with a vector of input trajectories $\mathbf{u}_k^{k+N}$ and parameter vector $\mathbf{p}$. This parameter vector contains for example system initial state $\mathbf{x}_{k_0}$, reference signal trajectories and some other possible parameters.

Offset free tracking

If we suppose a perfect model and no disturbances to a system with MPC, reference tracking with MPC controller with a minimum output movement cost function (2.43) is offset free by default. In practical applications this is rarely the case. There are many methods how to achieve offset free tracking of MPC even with model of a real system including model uncertainties and disturbances acting on the system. Two of these methods are widely used in practical applications [18]. The first approach is characterized by introducing an integral term into the cost function, which acts on tracking error. This is a similar strategy as the widely used method of achieving offset free tracking with standard PID controllers. This method has several disadvantages, as for example the necessity of implementation of an anti-windup mechanism, which makes the whole design procedure more complicated. The second approach is based on an assumption that there are virtual disturbance variables acting on the system. These virtual disturbances cover both model inaccuracy and actual disturbances. For practical implementation of this method it is usually assumed that the virtual disturbances are random walk processes. The mean values of these processes are used for the prediction over the whole prediction horizon. This approach to offset-free tracking is utilized using a system state observer. This second method is also known as Unknown Input Observer method [18].

The disturbance may be described by a general linear model and may be connected to the system in a number of ways. The simplest three ways of connecting the virtual disturbance to a linear system, which is described by state space equations (2.35) are representing the disturbance as if it is acting on the system output, system state or system input. The disturbance model can be described by the autonomous model of the form

$$\begin{aligned} \mathbf{x}_{d_{k+1}} &= \mathbf{A}_d \mathbf{x}_{d_k} \\ \mathbf{d}_k &= \mathbf{C}_d \mathbf{x}_{d_k} \ . \end{aligned} \tag{2.57}$$

If considering disturbance acting on the system output, the real system output is given by $\mathbf{y}_k = \hat{\mathbf{y}}_k + \mathbf{d}_k$. The system model is augmented by a state $\mathbf{x}_{d_k}$ and is of the form

$$\begin{aligned} \begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{x}_{d_{k+1}} \end{bmatrix} &= \begin{bmatrix} \mathbf{A} & 0 \\ 0 & \mathbf{A}_d \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{d_k} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \mathbf{u}_k \ , \\ \mathbf{y}_k &= \begin{bmatrix} \mathbf{C} & \mathbf{C}_d \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{d_k} \end{bmatrix} + \mathbf{D}\mathbf{u}_k \ . \end{aligned} \tag{2.58}$$

For the representation with the disturbance acting on the system state, the state equations are

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{x}_{d_{k+1}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{C}_d \\ 0 & \mathbf{A}_d \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{d_k} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \mathbf{u}_k \ , \qquad (2.59)$$

$$\mathbf{y}_k = \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{d_k} \end{bmatrix} + \mathbf{D}\mathbf{u}_k \ .$$

The third widely used interconnection is the case when the disturbance is connected to the system input

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{x}_{d_{k+1}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{C}_d \\ 0 & \mathbf{A}_d \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{d_k} \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \mathbf{u}_k \ , \qquad (2.60)$$

$$\mathbf{y}_k = \begin{bmatrix} \mathbf{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{x}_{d_k} \end{bmatrix} + \mathbf{D}\mathbf{u}_k \ .$$

Kalman filter has to be used to estimate the state of the augmented model. The virtual disturbance model structure influences the system performance, therefore, the choice of disturbance model may be seen as a tuning parameter of the MPC controller [2]. In practical applications a piecewise constant disturbance is often used.

2.4 Uncertainties and robustness

In order to design an MPC controller, there is a need for a system model. No matter how the system model is gained, there are some differences between the real system and the system model. These differences are called model uncertainties, or model/plant mismatch. The model uncertainties are caused by disturbances, inaccurate identification, incorrect model structure or by some system properties neglected during the modelling phase. In order to be able to control the real system accurately even with these uncertainties, it is important that the controlled system properties (stability, performance) are robust with respect to these uncertainties.

There are different origins of system uncertainties. Some parameter values are known only approximately or may be biased. The system structure or model order may be unknown at high frequencies. Sometimes it is useful to use a reduced order model for control system design.

There may be some system properties neglected during the system order reduction, these neglected properties may be represented as uncertainties as well. The sources of uncertainties may be grouped into two main categories. These are parametric uncertainties and dynamic uncertainties.

Parametric uncertainties occur when the model structure is known, but the exact values of system parameters are unknown. These values may be given as percentage deviations from mean value, or as intervals of possible parameter values. The uncertain parameter may be represented by a set of the form

$$\alpha_p = \overline{\alpha}(1 + r_\alpha \Delta) \ , \tag{2.61}$$

where $\overline{\alpha}$ is the mean parameter value, $r_\alpha = \frac{(\alpha_{max} - \alpha_{min})}{(\alpha_{max} + \alpha_{min})}$ is the relative uncertainty in parameter and $\Delta$is any real scalar satisfying $|\Delta| \leq 1$.

The dynamic uncertainties or so called frequency-dependent uncertainties are mainly caused by the fact that the dynamics of a system at higher frequencies are unknown or neglected. Any model of a real system contains this source of uncertainties. The dynamic uncertainty is less precise than parametric uncertainty, so it is more difficult to quantify. Frequency domain is well suited for dynamic uncertainty quantification. Complex perturbations, which are result of the quantification in frequency domain are normalized such that $\|\Delta(j\omega)\|_\infty \leq 1$.

The various sources of dynamic uncertainty may be lumped into a multiplicative uncertainty. For MIMO systems it is important to decide where the uncertainty should be represented. The system transfer function for the input uncertainty representation is represented as

$$\mathbf{G}_p(s) = \mathbf{G}(s)(\mathbf{I} + \mathbf{W}_I(s)\Delta_I(s)) \ , \tag{2.62}$$

where

$$\|\Delta_I(j\omega)\|_\infty \leq 1 \ \forall\omega \ .$$

The concept of uncertainties is common for SISO and MIMO systems. The right representation of uncertainties is very important especially for MIMO systems, because the system may be very sensitive to uncertainties in some directions, and insensitive to uncertainties in other directions. This property makes a big difference in system behaviour.

When considering MIMO systems, there are many sources of uncertainties in the system.

These uncertain perturbations may be pulled out to a block - diagonal matrix

$$\Delta = \text{diag}\{\Delta_i\} = \begin{bmatrix} \Delta_1 & & & \\ & \ddots & & \\ & & \Delta_i & \\ & & & \ddots \end{bmatrix},$$

where each Δ_i represents a specific source of uncertainty. This may be input uncertainty Δ_I, or parametric uncertainty δ_i, where δ_i is real. As the matrix Δ is diagonal, this description is called a structured uncertainty. There are different possible system configurations considering the uncertainties. In this thesis, the $M\Delta$-structure will be used, as it is useful representation for robustness analysis. Block diagram of this structure is depicted in Figure 2.4. The matrix **M**of this system representation is a transfer function from the uncertainty Δ output to its input.

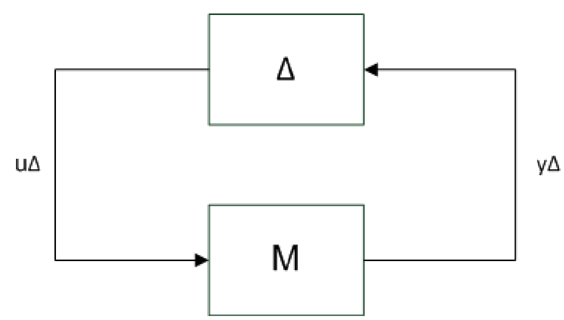

Figure 2.4: M Δ- structure for robust stability analysis

The main difference in uncertainty representation for MIMO systems comparing to SISO systems is the fact that the uncertainty has to be considered either at the input or at the output of a plant. There is no difference between an input and output uncertainties in SISO system, but because of the input and output directions in MIMO systems, it is important to represent the uncertainty properly. It is possible to transform an input uncertainty to output uncertainty and vice versa to have only one uncertainty matrix representing for all the system uncertainties. There are six different system configurations when using the unstructured uncertainty matrix. Four most common representations will be used in this thesis. The difference between these configurations is in the part of the system where the uncertainties are modeled (input or output) and in the form of uncertianty integration in the system (feedforward and feedback). The feedforward forms are the additive uncertainty, multiplicative input uncertainty and multiplicative output uncertainty with the following representations:

Additive uncertainty:

$$\prod_A: \ \mathbf{G}_p = \mathbf{G} + \mathbf{E}_A; \ \mathbf{E}_A = \mathbf{W}_A \Delta_A \ , \tag{2.63}$$

With a block diagram depicted in Figure 2.5.

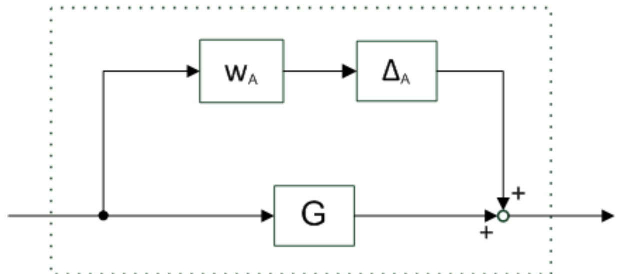

Figure 2.5: Block diagram of a system with additive uncertainty

Multiplicative input uncertainty

$$\prod_I: \ \mathbf{G}_p = \mathbf{G}(\mathbf{I} + \mathbf{E}_I); \ \mathbf{E}_I = \mathbf{W}_I \Delta_I \ , \tag{2.64}$$

with a block diagram depicted in Figure 2.6.

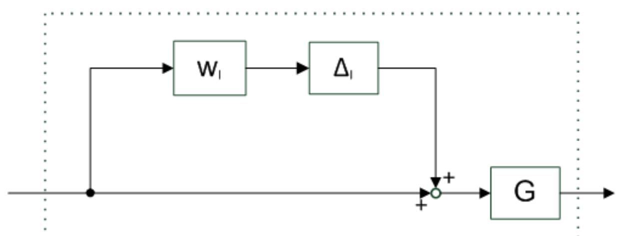

Figure 2.6: Block diagram of a system with multiplicative input uncertainty

Multiplicative output uncertainty

$$\prod_O: \ \mathbf{G}_p = (\mathbf{I} + \mathbf{E}_O)\mathbf{G}; \ \mathbf{E}_O = \mathbf{W}_O \Delta_O \ , \tag{2.65}$$

with a block diagram depicted in Figure 2.7.

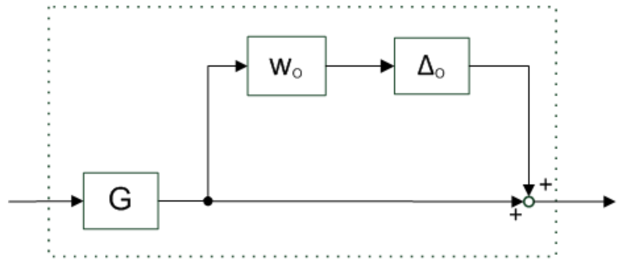

Figure 2.7: Block diagram of a system with multiplicative output uncertainty

The feedback form presented in this thesis will be the inverse additive uncertainty

$$\prod_{iA}: \ \mathbf{G}_p = \mathbf{G}(\mathbf{I}-\mathbf{E}_{iA}\mathbf{G})^{-1}; \ \mathbf{E}_{iA} = \mathbf{W}_{iA}\Delta_{iA} \ , \tag{2.66}$$

with a block diagram depicted in Figure 2.8.

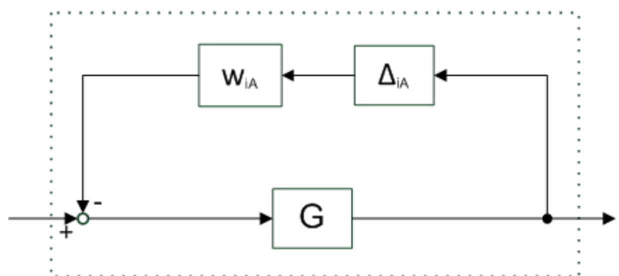

Figure 2.8: Block diagram of a system with inverse additive uncertainty

In all the above mentioned formulas Δ denotes the normalized perturbation and $\mathbf{E}$ the actual perturbation. Scalar weights w are used so that $\mathbf{E} = w\Delta = \Delta w$, but matrix weights may be used as well: $\mathbf{E} = \mathbf{W}_2\Delta\mathbf{W}_1$ where $\mathbf{W}_1$ and $\mathbf{W}_2$ are given transfer function matrices.

For SISO system it is easy to cumulate multiple sources of uncertainty into a single complex perturbation. The multiplicative form is often used. Similar approach is possible for MIMO system, but as mentioned earlier, it has to be decided whether the perturbation is at the input or the output of the plant. The output uncertainty is often less restrictive than input uncertainty in terms of control performance, so the first choice is to lump the uncertainty at the output. As an example a set of plants $\prod$ may be represented by multiplicative output uncertainty with a matrix weight $\mathbf{W}_O(s)$ using

$$\mathbf{G}_p = (\mathbf{I}+\mathbf{W}_O\Delta_O)\mathbf{G}, \ \|\Delta_O\|_\infty \leq 1 \ .$$

The uncertainty weight has to fulfil

$$\|\mathbf{W}_O(j\omega)\|_\infty \geq l_O(\omega)\ \forall\omega\ ,$$

where

$$l_o(\omega) = \max_{G_p \in \prod} \overline{\sigma}((\mathbf{G}_p - \mathbf{G})\mathbf{G}^{-1}(j\omega))\ .$$

This lumping of uncertainty is fine if the resulting uncertainty weight is at least less than one in the frequency range where the control is desired, and if robust stability and performance may be achieved. If these criteria are not fulfilled, then the uncertainty may be lumped at the input instead, using multiplicative input uncertainty with a matrix weight:

$$\mathbf{G}_p = \mathbf{G}(\mathbf{I} + \mathbf{W}_I\Delta_I),\ \|\Delta_I\|_\infty \leq 1\ ,$$

the uncertainty weight has to fulfil

$$\|\mathbf{W}_I(j\omega)\|_\infty \geq l_I(\omega)\ \forall\omega\ ,$$

where

$$l_I(\omega) = \max_{G_p \in \prod} \overline{\sigma}(\mathbf{G}^{-1}(\mathbf{G}_p - \mathbf{G})(j\omega))\ .$$

Neither of these two above mentioned lumpings of uncertainty may work for some systems. This is caused by the fact, that a perturbation cannot be generally shifted from one location in the plant to another location, for example from the input to the output, without introducing candidate plants which were not present in the original set. This is in particular important in the case of ill-conditioned systems. When moving a perturbation from one location in a MIMO system to another location, it has to be multiplied by the condition number $\gamma(\mathbf{G})$. When the true uncertainty, represented as unstructured input uncertainty in the form $\mathbf{G}_p = \mathbf{G}(\mathbf{I} + \mathbf{E}_I)$ with the magnitude of multiplicative input uncertainty

$$l_I(\omega) = \max_{G_p \in \prod} \overline{\sigma}(\mathbf{G}^{-1}(\mathbf{G}_p - \mathbf{G}) = \max_{E_I} \overline{\sigma}(\mathbf{E}_I)$$

is desired to be represented as multiplicative output uncertainty in the form

$$l_o(\omega) = \max_{E_I} \overline{\sigma}((\mathbf{G}_p - \mathbf{G})\mathbf{G}^{-1}) = \max_{E_I} \overline{\sigma}(\mathbf{G}\mathbf{E}_I\mathbf{G}^{-1})\ ,$$

which is much larger than $l_I(\omega)$ if the condition number of the plant is large. If we write $\mathbf{E}_I = \mathbf{W}_I\Delta_I$ with any $\Delta_I(j\omega)$ satisfying $\overline{\sigma}(\Delta_I(j\omega)) \leq 1,\ \forall\omega$ allowed, then at a given frequency

$$l_O(\omega) = \|\mathbf{W}_I\|_\infty \max_{\Delta_I} \overline{\sigma}(\mathbf{G}\Delta_I\mathbf{G}^{-1}) = \|\mathbf{W}_I(j\omega)\|_\infty \gamma(\mathbf{G}(j\omega)) \ .$$

Similar situation may occur even when using diagonal uncertainty ($\mathbf{E}_I$ diagonal). If the RGA elements of the plant are large, then the elements in $\mathbf{G}\mathbf{E}_I\mathbf{G}^{-1}$ are much larger than the elements of $\mathbf{E}_I$, which makes it difficult to move uncertainty from the system input to the output. To conclude this section it is important to mention, that even though it is practical to lump all model uncertainties into a single perturbation either at input or output, in the case of ill-conditioned plants it is sometimes necessary to represent the uncertainty as it occurs physically, thereby generating several perturbations. When representing uncertainty of a plant with unstable poles, one of the inverse forms of uncertainty representation should be used.

Very similar approach to the uncertainty representation may be used for system with diagonal uncertainties. The diagonal uncertainty originates from independent scalar uncertainty in each input channel. Diagonal uncertainty is represented by a complex diagonal matrix

$$\Delta(s) = \text{diag}\,\{\delta_i(s)\}\,;\ |\delta_i(j\omega)| \leq 1,\ \forall\omega\ .$$

Diagonal uncertainty usually arises from a consideration of uncertainty or neglected dynamics in the individual input channels (actuators) or in the individual output channels (sensors). As an example, a diagonal input uncertainty may be described as

$$\mathbf{G}_p(s) = \mathbf{G}(\mathbf{I} + \mathbf{W}_I\Delta_I);\ \Delta_I = \text{diag}\,\{\delta_i\}\,,\ \mathbf{W}_I = \text{diag}\,\{w_{Ii}\}\ .$$

The uncertainty in each input or output channel is normaly represented using a simple weight in the form

$$w(s) = \frac{\tau s + r_0}{(\tau/r_\infty)s + 1}\ ,$$

where r_0 is the relative uncertainty at steady-state, $1/\tau$ is the frequency where the relative uncertainty approximately reaches 100%, and r_∞ is the magnitude of the weight at higher frequencies. Typically, the uncertainty $|w|$, associated with each input, is at least 10% at steady state, and it increases at higher frequencies to account for neglected or uncertain dynamics [1].

2.4.1 Robust stability

To ensure internal stability of a plant with uncertainties for all the possible perturbations, the system has to be robustly stable. At this point, we will consider only the robust stability for plants with unstructured uncertainties, which means that $\Delta(s)$ is allowed to be any complex transfer function matrix satisfying $\|\Delta\|_\infty \leq 1$. For robust stability analysis it is convenient to use the $M\,\Delta$ structure from Figure 2.4, as it is possible to easily derive the transfer function from the output of the uncertainty to its input. It is assumed that the nominal system $\mathbf{M}(s)$ is stable and the perturbations $\Delta(s)$ are stable as well. Then the $M\,\Delta$ - system is stable for all perturbations Δ satisfying $\|\Delta\|_\infty \leq 1$ if and only if

$$\bar{\sigma}(\mathbf{M}(j\omega)) < 1\ \forall\omega \quad \Leftrightarrow \quad \|\mathbf{M}\|_\infty < 1\ , \tag{2.67}$$

which may be rewritten as

$$\text{Robust Stability} \ \Leftrightarrow\ \bar{\sigma}(\mathbf{M}(j\omega))\bar{\sigma}(\Delta(j\omega)) < 1,\ \ \forall\omega,\ \forall\Delta\ . \tag{2.68}$$

This may be seen as a MIMO version of the well known Small gain theorem[1], because the sufficiency of (2.68) follows from choosing $\mathbf{L} = \Delta\mathbf{M}$ in the Small gain theorem. At this point it is important to justify the use of H_∞ norm. It is used to analyze robust stability, because the stability condition in (2.68) is both necessary and sufficient. If the H_2 norm would be used at this point, the stability condition would neither be necessary nor sufficient [1].

The matrix M for the $M\Delta$ structure representation of a system may be derived for all the uncertainty representations presented earlier in section 2.4.1. The uncertainty weight may be a scalar or a matrix. In the scalar case we can write

$$\mathbf{E} = w\Delta = \Delta w,\ \|\Delta\|_\infty \leq 1\ ,$$

and in the matrix case

$$\mathbf{E} = \mathbf{W}_2\Delta\mathbf{W}_1,\ \|\Delta\|_\infty \leq 1\ .$$

The matrix $\mathbf{M}$ depicted in Figure 2.4 is determined, so that the perturbation of the system is isolated in terms of transfer function $\mathbf{M}$ from the output to the input of the perturbation Δ. This

[1] Small gain theorem for stable loop transfer function is expressed as $\|\mathbf{L}(j\omega)\| < 1\ \forall\omega$, where $\|\mathbf{L}\|$ denotes any matrix norm satisfying $\|\mathbf{AB}\| \leq \|\mathbf{A}\| \cdot \|\mathbf{B}\|$

may be written as $\mathbf{M} = \mathbf{W}_1\mathbf{M}_0\mathbf{W}_2$ in the case of matrix representation of the uncertainty weights, and in a simmilar manner $\mathbf{M} = w\mathbf{M}_0$ for the scalar case. $\mathbf{M}_0$ may be derived for all the different mentioned uncertainty representations.

As it will be useful in later stage of this thesis, a modified $M\Delta$ structure will be used. The weighting matrix $\mathbf{W}$ will be pulled out of the matrix $\mathbf{M}$. This representation of a system, displayed in Figure 2.9 enables us to find the robust stability condition of different uncertainty representations in terms of weighting matrix $\mathbf{W}$. The matrices $\mathbf{W}$ for different uncertainty representations will be presented in this paragraph as well.

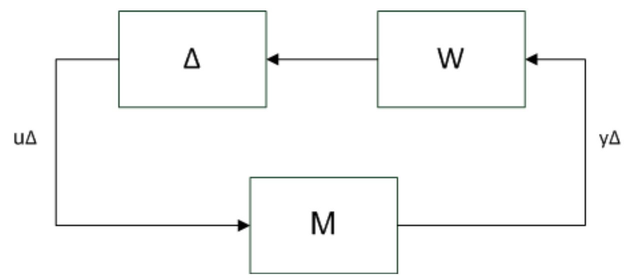

Figure 2.9: $M\Delta$-structure with weighting matrix pulled out from $\mathbf{M}$

The weighting matrices for different uncertainty representations will be derived at this point. The derived conditions will be used in section 3.2 to set the robust stability conditions for the individual uncertainty representations..
For the additive uncertainty with $\mathbf{G}_p = \mathbf{G}_0 + \Delta\mathbf{W}_A$ we get

$$\mathbf{M}_0 = \mathbf{K}(\mathbf{I} + \mathbf{G}_0\mathbf{K})^{-1} = \mathbf{KS} \; . \tag{2.69}$$

If the uncertainty is represented as ε, the perturbed plant is $\mathbf{G}_p = (1+\varepsilon)\mathbf{G}_0$

$$(1+\varepsilon)\mathbf{G}_0 = \mathbf{G}_0 + \Delta\mathbf{W}_A \; ,$$

$$\varepsilon\mathbf{G}_0 = \Delta\mathbf{W}_A \; ,$$

considering $||\Delta||_\infty \leq 1$, the weighting function has to fulfil

$$||\varepsilon\mathbf{G}_0||_\infty \leq ||\mathbf{W}_A||_\infty \; .$$

In order for the system to be robustly stable, the actual frequency characteristics of the appropriate transfer function $\mathbf{G}_A$ has to fulfil the condition

$$||\mathbf{G}_A||_\infty \leq \frac{1}{||\mathbf{W}_A||_\infty} \; . \tag{2.70}$$

For the multiplicative input uncertainty with $\mathbf{G}_p = \mathbf{G}_0(\mathbf{I} + \Delta\mathbf{W}_{MI})$ we get

$$\mathbf{M}_0 = \mathbf{K}(\mathbf{I} + \mathbf{G}_0\mathbf{K})^{-1}\mathbf{G}_0 = \mathbf{T}_I \; . \tag{2.71}$$

Representing the uncertainty as ε, the perturbed plant is $\mathbf{G}_p = (1+\varepsilon)\mathbf{G}_0$ and the Robust stability condition becomes

$$(1+\varepsilon)\mathbf{G}_0 = \mathbf{G}_0(\mathbf{I} + \Delta\mathbf{W}_{MI}) \; ,$$

considering $\|\Delta\|_\infty \leq 1$, the weighting function has to fulfill

$$\|\varepsilon\|_\infty \leq \|\mathbf{W}_{MI}\|_\infty \; ,$$

which yields the robust stability condition for the frequency characteristics of the transfer function for the multiplicative input uncertainty $\mathbf{G}_{MI}$

$$\|\mathbf{G}_{MI}\|_\infty \leq \frac{1}{\|\mathbf{W}_{MI}\|_\infty} \; . \tag{2.72}$$

For the multiplicative output uncertainty with $\mathbf{G}_p = (\mathbf{I} + \Delta\mathbf{W}_1)\mathbf{G}_0$ we get

$$\mathbf{M}_0 = \mathbf{G}_0\mathbf{K}(\mathbf{I} + \mathbf{G}_0\mathbf{K})^{-1} = \mathbf{T} \; , \tag{2.73}$$

if we represent the uncertainty as ε, the perturbed plant is $\mathbf{G}_p = (1+\varepsilon)\mathbf{G}_0$ and

$$(1+\varepsilon)\mathbf{G}_0 = (\mathbf{I} + \Delta\mathbf{W}_{MO})\mathbf{G}_0 \; ,$$

considering $\|\Delta\|_\infty \leq 1$, the weighting function $\mathbf{W}_{MO}$has to fulfill

$$\|\varepsilon\|_\infty \leq \|\mathbf{W}_{MO}\|_\infty \; .$$

Robust stability condition for the frequency characteristics of the plant with multiplicative output uncertainty representation $\mathbf{G}_{MO}$ is expressed as

$$\|\mathbf{G}_{MO}\|_\infty \leq \frac{1}{\|\mathbf{W}_O\|_\infty} \; . \tag{2.74}$$

And for the inverse additive uncertainty with $\mathbf{G}_p = \mathbf{G}_\mathbf{0}(\mathbf{I} - \Delta\mathbf{W}_1\mathbf{G}_0)^{-1}$ we get

$$\mathbf{M}_0 = (\mathbf{I} + \mathbf{G}_0\mathbf{K})^{-1}\mathbf{G}_0 = \mathbf{S}\mathbf{G}_0 \; , \tag{2.75}$$

if we represent the uncertainty as ε, the perturbed plant is $\mathbf{G}_p = (1+\varepsilon)\mathbf{G}_0$, using this identity, we get

$$(1+\varepsilon)\mathbf{G}_0 = \mathbf{G}_0(\mathbf{I}-\Delta\mathbf{W}_{IA}\mathbf{G}_0)^{-1} \ ,$$

$$1+\varepsilon = (\mathbf{I}-\Delta\mathbf{W}_{IA}\mathbf{G}_0)^{-1} \ ,$$

$$(1+\varepsilon)(\mathbf{I}-\Delta\mathbf{W}_{IA}\mathbf{G}_0) = 1 \ ,$$

$$1-\Delta\mathbf{W}_{IA}\mathbf{G}_0+\varepsilon-\varepsilon\Delta\mathbf{W}_{IA}\mathbf{G}_0 = 1 \ ,$$

considering $\|\Delta\|_\infty \leq 1$, the condition for the weighting function $\mathbf{W}_{IA}$becomes

$$\left\| \frac{\varepsilon}{(1+\varepsilon)\mathbf{G}_0} \right\|_\infty \leq \|\mathbf{W}_{IA}\|_\infty \ . \tag{2.76}$$

In this case it is necessary to consider the smalest singular value of G_0, as we need all possible perturbed plants to be placed below the weight $\mathbf{W}_{IA}$. For the system to be robustly stable the transfer function $\mathbf{G}_{IA}$ for he $M\Delta$system representation has to fulfil the robust stability condition

$$\|\mathbf{G}_{IA}\|_\infty \leq \frac{1}{\|\mathbf{W}_1\|_\infty} \ .$$

The derived formulas for evaluating the weighting functions of different uncertainty representations will be used later in this thesis to derive stability conditions.

Chapter 3

Predictive control of ill-conditioned plants

There are many ill-conditioned systems that need to be controlled in practical applications. The conventional control methods use the input and output pairing choice as the strongest tool for dealing with this type of systems. Simply by selecting the best combination of inputs and outputs of the system the combination that is easiest to control and delivers the required results is chosen for controller design. The conventional control methods for MIMO systems which are described in section 2.3 are usually used. The RGA and Gramian based interaction measures are useful when making the input-output pairs selection for a system. RGA is a conventional method known since the 1966 [5], whereas the Gramian based methods are recent methods used in many modifications since the end of 1990's [11, 10]. These tools may be used when designing a Model Predictive Controller for ill-conditioned systems as well. They are useful when the system inputs and outputs are being chosen. The advantage of using MPC for ill-conditioned systems control is the fact, that there are some additional possibilities of how to cope with the ill-conditioning of the system in addition to the suitable input-output pairs selection, which is done when the conventional control techniques are employed. The aim of this thesis is to introduce additional modifications to the MPC algorithm that make use of information gained from the system conditioning analysis.

There were different techniques used at some previous MPC applications for ill-conditioned systems. Although these techniques were proved to be suitable for ill-conditioned systems, the approach used for MPC design did not really consider the system directions. Only a side-step to avoid some of the disadvantages and undesirable behaviour of ill-conditioned systems has been made. Two techniques used for MPC control of ill-conditioned plants were presented in [17]. The first technique is based on avoiding control of the outputs corresponding to the small singular values of a system. This elimination of system outputs is done by appropriate setting of cost function tuning parameters, namely the elements of weighting matrix **Q** in cost function (2.43). The elements corresponding to the outputs that need to be eliminated are set to zero. Although this is one possibility of ill-conditioned systems control, this approach is restrictive in terms of achieved reference tracking. It will be shown in this thesis that behaviour of MPC may be improved by changing the weighting matrices of MPC cost function without a need to completely eliminate a system output. The second method presented in [17] uses the fact, that it is not always necessary to track a reference signal precisely. The proposed method changes the aim of a controller from particular trajectory tracking into keeping the output in certain regions. In praxis this is applicable to a large spectra of processes [17]. The conventional MPC cost function has to be changed slightly when employing this method. The reference values **r** in the cost function (2.41) become regions. The individual regions are defined by a minimum and maximum values of allowable output. In order for an input to lie within a certain region $\mathbf{y}_{t+k|t}$ the output value has to fulfil the inequality $\mathbf{y}_{lower} \leq \mathbf{y}_{t+k|t} \leq \mathbf{y}_{upper}$, where $\mathbf{y}_{lower}$ is the lower region limit and $\mathbf{y}_{upper}$ is the upper region limit. The modified cost function is then

$$J\left(\mathbf{u}|\mathbf{x}(t_0), t_0\right) = \sum_{k=0}^{N_y-1} \left[\mathbf{z}_{t+k|t}\mathbf{Q}\mathbf{z}_{t+k|t} + \Delta\mathbf{u}_{t+k}\mathbf{R}\Delta\mathbf{u}_{t+k}\right] \tag{3.1}$$

where $\mathbf{z}_{t+k|t} \geq 0$; $\mathbf{z}_{t+k|t} = \mathbf{y}_{t+k|t} - \mathbf{y}_{upper}$ for $\mathbf{y}_{t+k|t} > \mathbf{y}_{upper}$;

$\mathbf{z}_{t+k|t} = \mathbf{y}_{lower} - \mathbf{y}_{t+k|t}$ for $\mathbf{y}_{t+k|t} - \mathbf{y}_{upper}$; $\mathbf{z}_{t+k|t} = 0$ for $\mathbf{y}_{lower} \leq \mathbf{y}_{t+k|t} \leq \mathbf{y}_{upper}$. No control actions are taken when the output lies inside the desired region, which is when $\mathbf{z}_{t+k|t} = 0$. If an output violates a desired region, the control objective in the MPC regulator will activate and push it back to the desired region [17]. This approach to ill-conditioned systems MPC control has been proved as a good performing alternative to the approach with output avoiding of MIMO systems presented in the same article [17]. No practical examples where uncertainty and model mismatch have been taken into account were given in [17]. This approach to MPC has been used previously in different applications, it is known as Model Predictive Range Control [23].

3.1 Extended MPC algorithm

There will be two modifications of the conventional MPC algorithm presented in this thesis. These methods were designed to improve the MPC behaviour when an ill-conditioned system is controlled. Not only the system reference tracking abilities, but also the system robust stability will be concerned. The two modifications of MPC will be proposed first, the theoretical results will be demonstrated on a simplified model of a practical ill-conditioned system later on. Distillation column has been chosen as a model of an ill-conditioned system. The analysis of this MIMO system will be performed first to show the system properties and to show that the system is ill-conditioned. This analysis will be followed by a controller design based on the proposed modified MPC for ill-conditioned plants. Robust stability of the resulting feedback system will be checked and compared with results of conventional MPC applied to the same system.

3.1.1 Input absolute value penalization

Properties of ill-conditioned systems described in chapter 2.1 cause some problems if controlled by conventional controllers. The fact that the system has a large gain for some input directions and a small gain for other input directions causes difficulties when the system tracks a reference or when there is a disturbance to be rejected. Problems occur namely when the reference tracking or disturbance rejection require a change in a direction corresponding to the smallest singular value $\underline{\sigma}(\mathbf{G})$ of the system. Although the required change of output may not be large, the system is not able to reach the reference, as the system gain in the desired direction is very small. In praxis, the controller may try to reach the reference by increasing the system input value. This may lead to unconstrained progression in the system input, which is undesirable as the high input has almost no influence on the change of an output. The basic idea of avoiding such situation in MPC design is to penalize the absolute value of system input. This modification of MPC would not improve the quality of reference tracking if the system is not able to track the reference in some directions, but it limits the unconstrained rise of system input.

At this point only the modifications of the standard MPC algorithm will be explained. There are some additional steps necessary in the control system design when this method is applied, but only its principle will be presented at this point. The whole design procedure will be described in following sections.

The standard cost function for minimum movement controller

$$J\left(\mathbf{u}_k^{k+N}|\mathbf{x}_k,k\right) = \mathbf{e}_k^T\mathbf{Q}\mathbf{e}_k + \Delta\mathbf{u}_k^T\mathbf{R}\Delta\mathbf{u}_k \tag{3.2}$$

where $\mathbf{e}_k = (\mathbf{r}_k - \mathbf{y}_k)$ is extended by a cost function member penalizing the input value. The resulting cost function is

$$J\left(\mathbf{u}_\mathbf{k}^{\mathbf{k+N}}|\mathbf{x}_k,k\right) = \mathbf{e}_k^T\mathbf{Q}\mathbf{e}_k + \Delta\mathbf{u}_k^T\mathbf{R}\Delta\mathbf{u}_k + \mathbf{u}_k^T\mathbf{R}_u\mathbf{u}_k \ , \tag{3.3}$$

where $\Delta\mathbf{u}_k = \mathbf{u}_k - \mathbf{u}_{k-1}$. The original cost function from equation (3.2) is extended by a term penalizing the input value by a diagonal weighting matrix $\mathbf{R}_u$. This matrix now may be seen as an additional MPC tuning tool. By appropriate setting of its elements it is possible to limit the rise of the system input.The setting of this weighting matrix has to be done with a care, as large penalization may affect the system behaviour in an undesired way.

3.1.2 Input movement in weak directions penalization

The second proposed method uses deeper knowledge about the system properties, therefore, it requires a system analysis before it can be accomplished. As describedin section 2.2.1,it is possible to find the system input directions which correspond to the small singular values of a system using the Singular Value Decomposition. A large input value in a weak input direction has a small effect on the system output and therefore it is undesirable. The idea of a cost function modification is to penalize the controller movement in a weak input direction of the system. To perform this modification, the weak directions have to be found first. The results of system gain matrix SVD decomposition will be used at this point. The matrix Σ of SVD has the system singular values on its diagonal in the descending order. The system has the largest gain in the direction corresponding to the largest singular value, which is the first element on the diagonal of Σ. A system singular value σ_i may be compared to the largest singular value $\overline{\sigma}$. If the ratio between the largest singular value $\overline{\sigma}$ and singular value σ_i is larger than a certain threshold, the direction corresponding to the singular value σ_i may be considered as a system weak input direction. System condition number $\gamma(\mathbf{G})$ has been described in section 2.2.3. It is a ratio between the largest and smallest singular value and it is a widely-used analysis tool for system conditioning. Based on the concept of condition number the threshold for weak directions of the system may be determined. Although the condition number is a measure of ill-conditioning of a system, there is no exact value of this indicator that would be considered as a threshold value between well-conditioned and ill-conditioned systems. In most cases the systems with condition number much larger than 10 are considered as ill-conditioned [1]. All the system singular values are compared to the largest singular value during the MPC design procedure. If the ratio between these two numbers is larger than $\gamma(\mathbf{G}) = 10$, the input direction corresponding to this singular value is considered as a weak input direction. The input directions are the column vectors of matrix $\mathbf{V}_u$ of the system SVD decomposition. As the singular values are organized on the diagonal of Σ in a descending order, it would be possible to evaluate the ratio between $\overline{\sigma}$ and σ_i only until the first singular value that violates the threshold is found. All the column vectors of matrix $\mathbf{V}_u$ to the right from this vector are the weak input direction as well. The incorporation of weak input directions penalization requires some modification of the minimum movement cost function (2.43). Using the system Singular Value Decomposition $\mathbf{G} = \mathbf{U}_y \Sigma \mathbf{V}_u^T$, where the input vectors are columns of matrix $\mathbf{V}_u$ a change of the system output may be written as

$$\Delta \mathbf{y} = \mathbf{G} \Delta \mathbf{u}\,, \tag{3.4}$$

using the SVD matrices this is expressed as

$$\Delta \mathbf{y} = \mathbf{U}_y \Sigma \mathbf{V}_u^T \Delta \mathbf{u}\,. \tag{3.5}$$

The system input change including directions may be denoted $\Delta \mathbf{u}_{dir}$ and expressed as

$$\Delta \mathbf{u}_{dir} = \mathbf{V}_u^T \Delta \mathbf{u}\,. \tag{3.6}$$

To include this term into the quadratic cost function, it has to be written as

$$\Delta \mathbf{u}_{dir}^T \Delta \mathbf{u}_{dir} = (\mathbf{V}_u^T \Delta \mathbf{u})^T (\mathbf{V}_u^T \Delta \mathbf{u}) = \Delta \mathbf{u}^T \mathbf{V}_u \mathbf{V}_u^T \Delta \mathbf{u}\,. \tag{3.7}$$

The vectors $\mathbf{v}_{u_{weak}}$, corresponding to the small system singular values, are chosen from the input rotation matrix $\mathbf{V}_u^T$. A matrix $\mathbf{H}_{wd}$ is constructed using these weak direction vectors and a weighting factor for weak directions penalization α_{wd} as

$$\mathbf{H}_{wd} = \mathbf{I}_N \otimes \left(\sum_{i=1}^{k_{wd}} \alpha_{wd_i} \mathbf{v}_i \mathbf{v}_i^t \right)\,, \tag{3.8}$$

where $\otimes$ is a Kronecker multiplication operator, and N is the MPC prediction horizon length. The Kronecker product is made in order for the term to fit the sizes of other cost function terms. Larger penalizing terms α_{wd_i} mean larger penalization of input increment in the system weak input directions. The cost function including the term for weak input direction movement penalization is of the form

$$J\left(\mathbf{u_k^{k+N}}|\mathbf{x}_k, k\right) = \mathbf{e}_k^T \mathbf{Q} \mathbf{e}_k + \Delta\mathbf{u}_k^T \mathbf{R} \Delta\mathbf{u}_k + \Delta\mathbf{u}_k^T \mathbf{H}_{wd} \Delta\mathbf{u}_k \; . \tag{3.9}$$

If the cost function (3.9) is minimized over $\Delta\mathbf{u}$, using the output equation (2.39), optimized input series in terms of $\Delta\mathbf{u}$ is

$$\Delta\mathbf{u}_k^{\star k+N} = (\bar{\mathbf{H}}^T \mathbf{Q} \bar{\mathbf{H}} + \mathbf{R} + \mathbf{H}_{wd})^{-1} (\bar{\mathbf{H}}^T \mathbf{Q} \bar{\mathbf{P}}) \; .$$

The performance and stability of this augmented MPC controller will be shown on a model of ill-conditioned distillation column in the following chapter.

3.2 Example - distillation column control

In this section the methods proposed in section 3.1 will be applied to a model of a real system, simulations will be performed on the model and system robust stability will be verified.

3.2.1 System description

To illustrate the obtained results of ill-conditioned process control, a model of distillation column will be used. There are more reasons for choosing this process as a model for simulations, although there are many other ill-conditioned systems in all branches of control systems application. Distillation is one of the most common unit operations in the chemical industry. Modelling, identification and control of the distillation process have been paid much attention in the literature and represents one of the most popular examples of ill-conditioned systems within the control comunity [15]. The distillation column is used as an example of an ill-conditioned system in several analyses [15, 1, 3, 17]. Distillation column control has been found challenging due to the fact that the process often shows strong interactions, nonlinearities and is ill-conditioned. Another reason for popularity of using distillation process as an exemplary plant within control community is the fact, that the potential gain of improving the quality of its control is large, as distillation is highly energy consuming [15]. There are many different distillation column models described in the literature. These models differ in system model order, number and choice of manipulated variables and linearity of the model. A linearized and scaled second order model (presented by Skogestad et al. [3]) has been chosen for the purpose of simulations and testing of the proposed methods of Model Predictive Control of an ill-conditioned plant in this thesis.

The objective of the distillation column, which is schematically illustrated in Figure 3.1, is to split the feed F, which is a mixture of a light and a heavy component, into a distillate product D, which contains most of the light component, and a bottom product B, which contains most of the heavy component.

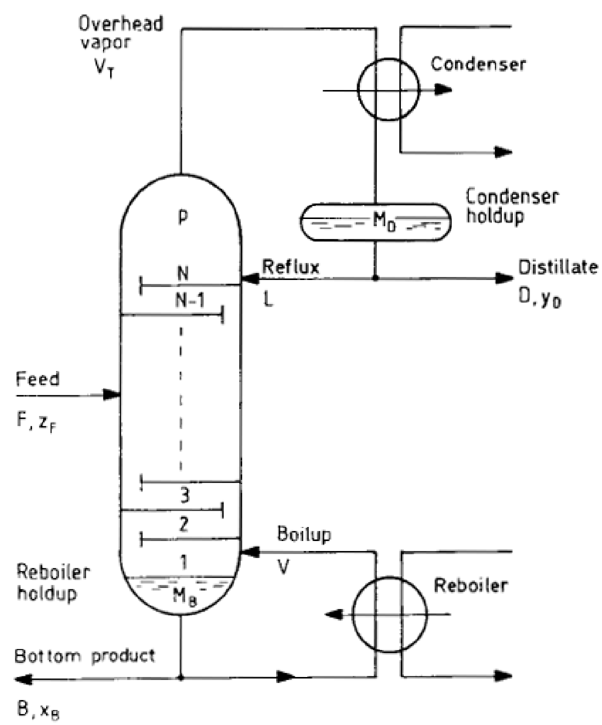

Figure 3.1: Schematic illustration of the distillation column [3]

The distillation column in Figure 3.1 has five controlled variables [3]:

1. vapor holdup (expressed by the pressure p)
2. liquid holdup in the accumulator (M_D)

3. liquid holdup in the column base (M_B)

4. top composition (y_D)

5. bottom composition (x_B)

and five manipulated inputs:

1. distillate flow (D)

2. bottom flow (B)

3. reflux (L)

4. boilup (V) (controlled indirectly by the reboiler duty)

5. overhead vapor (V_T) (controlled indirectly by the condenser duty)

There are different possible control configurations of the system, which differ in the chosen inputs and outputs for the distillation process control. The so-called LV-configuration will be used for composition control in this chapter. This configuration uses changes in internal flows of the distillation column by simultaneous changes in reflux L and boilup V. A linearized model, where the overhead composition is to be controlled at $y_o = 0.99$ and the bottom composition at $x_B = 0.01$ will be used. This choice is often made since reflux (L) and boilup (V) have an immediate effect on the product compositions [3]. A linearized model, which assumes that the dynamics may be approximated by a first-order response with time constant $\tau = 75$min results in the following scaled transfer function

$$\begin{pmatrix} dy_D \\ dx_B \end{pmatrix} = \mathbf{G}_{LV} \begin{pmatrix} dL \\ dV \end{pmatrix}, \ \mathbf{G}_{LV} = \frac{1}{\tau s + 1} \begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix}. \tag{3.10}$$

This is admittedly a very crude model of this strongly nonlinear plant, but the model is simple and displays important features of the distillation column behaviour [3].

3.2.2 Analysis of the system model

The distillation is a complex process, which needs a deep knowledge of the ongoing chemical processes inside the distillation column. If we were given all the system inputs and outputs of the plant in order to choose some of them to control the distillation process, the RGA or Gramian based methods would be helpful for the decision.

The *LV*-configuration is an approved system configuration, chosen based on the practical experience with the distillation column [1]. The following relationships must hold during the ongoing distillation process:

$$dV = dV_T, \;\; dD = -dB = dV - dL \, . \tag{3.11}$$

Irrespective of the control configuration, the two operating variables corresponding to the high and low plant gain are the external flows and the internal flows. The external flows are changed by making changes in the product flows B and D. The internal flows are changed by making simultaneous changes in reflux L and boilup V, while keeping D and B constant [3].
To analyze the properties of the MIMO system, described by transfer function (3.10), an SVD decomposition will be done. The decomposed system is of the form

$$\mathbf{G}_{LV} = \underbrace{\begin{bmatrix} -0.6246 & -0.7809 \\ -0.7809 & 0.6246 \end{bmatrix}}_{\mathbf{U}_y} \underbrace{\begin{bmatrix} 1.9721 & 0 \\ 0 & 0.0139 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} -0.7066 & -0.7077 \\ 0.7077 & -0.7066 \end{bmatrix}^T}_{\mathbf{V}_u^T} . \tag{3.12}$$

The *LV* configuration condition number is

$$\gamma_{LV}(\mathbf{G}_{LV}) = \frac{1.9721}{0.0139} = 141.732 \, . \tag{3.13}$$

The Singular value decomposition is not useful only for analysis of ill-conditioning of a system in terms of condition number. It also delivers important information on the system behaviour. From the input and output matrices $\mathbf{V}_u$ and $\mathbf{U}_y$ it is possible to decide what effect on the system behaviour have the different input and output directions. The input direction with highest gain is the first column of the matrix $\mathbf{V}_u$, which is in the case of distillation column $\bar{\mathbf{v}}_u = \begin{pmatrix} -0.7066 \\ 0.7077 \end{pmatrix} = \begin{pmatrix} dL \\ dV \end{pmatrix}$. This physically corresponds to the largest possible change in external flows D and B as $dB = dL - dV$. From the corresponding output direction vector $\bar{\mathbf{u}}_y = \begin{pmatrix} -0.6246 \\ -0.7809 \end{pmatrix}$ it is possible to see, that the highest singular value of the system corresponds to a movement of the outputs in the same direction, which means that the average composition $(y_D + x_B)$ is mainly affected. On the other hand, the smallest plant singular value $\underline{\sigma}(G_{LV})$ is obtained for inputs in direction $\underline{\mathbf{v}_u} = \begin{pmatrix} -0.7077 \\ -0.7066 \end{pmatrix} = \begin{pmatrix} dL \\ dV \end{pmatrix}$, which physically corresponds to changing internal flows only. The corresponding output vector $\underline{\mathbf{u}_y} = \begin{pmatrix} -0.7809 \\ 0.6246 \end{pmatrix}$ has the effect of moving the outputs in different directions. The physical interpretation is that it takes a large control action to make both products purer simultaneously [17]. This analysis of strong and weak input and output directions is very important, as it will be used for the distillation column MPC design. A high value of condition number $\gamma_{LV}(\mathbf{G}_{LV})$ by itself does not necessarily mean that the system is ill-conditioned. If a system was diagonal with a big difference between the largest and smallest singular value, the plant gain would strongly depend on input direction, but there would be no interactions in the system. It is therefore useful to make further analysis when a system that is likely to have interactions is to be controlled.

Although the choice of inputs and outputs of the system is not in focus of this work, it is useful to evaluate the RGA, as it confirms the assumption of ill-conditioning of the system that has been made using the SVD decomposition.
RGA of the distillation column is

$$\text{RGA}(\mathbf{G}_{LV}) \triangleq \mathbf{G}_{LV} \times (\mathbf{G}_{LV}^{-1})^T = \begin{pmatrix} 35.0688 & -34.0688 \\ -34.0688 & 35.0688 \end{pmatrix} . \tag{3.14}$$

Relative Gain Array is frequency dependent, the above evaluated RGA is calculated at the system steady state. According to Chapter 3 in Skogestad et al. [1], pairing of negative RGA elements should be avoided if possible. As can be seen from equation (3.14), the off-diagonal pairing of the distillation column LV configuration should be avoided. Pairing of RGA elements which are close to 1 is desirable. MPC does not require inputs and outputs pairing, but as the RGA elements on the diagonal are much higher than 1, it is obvious, that control of this plant is fundamentally difficult due to the strong interactions and sensitivity to uncertainty. To make a comprehensive system analysis, the RGA number may be evaluated for both possible diagonal pairings. Pairings with small RGA number are preferred.

$$\text{RGAnumber}_{\text{diag}} = \left\| \Lambda(\mathbf{G}_{LV}) - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\|_{sum} = 136.3 \; ,$$

$$\text{RGAnumber}_{\text{offdiag}} = \left\| \Lambda(\mathbf{G}_{LV}) - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\|_{sum} = 140.3\ .$$

Although there is not a huge difference between the diagonal and off-diagonal RGA number value, the diagonal pairing is preferred, as the value is smaller. This has confirmed our previous assumption to use a diagonal control of the distillation column.

System response analysis

Although it is not necessary to examine the open loop system behaviour for the controller design, it is illustrative to show how the ill-conditioned system responds to input signals in different directions. The following simulations are direct consequences of the system directionality shown by SVD. One possibility how to show ill-conditioning of the distillation column is to make a graphical analysis described in section 2.2. The input space is shown in the left plot of Figure 3.2, the system inputs form a unit circle, so all the system directions are fed equivalently. The output space in the right plot of Figure 3.2 shows the ill - conditioning of the system. The output space, where the system outputs y_1 and y_2 are plotted in face of each other has a shape of an ellipse, which means that it has a large gain in the direction of major ellipse axis and a small gain in the direction of minor ellipse axis. From the output space ellipse we can also conclude that the system is linearized, as the ellipse would be bent if the system was non-linear.

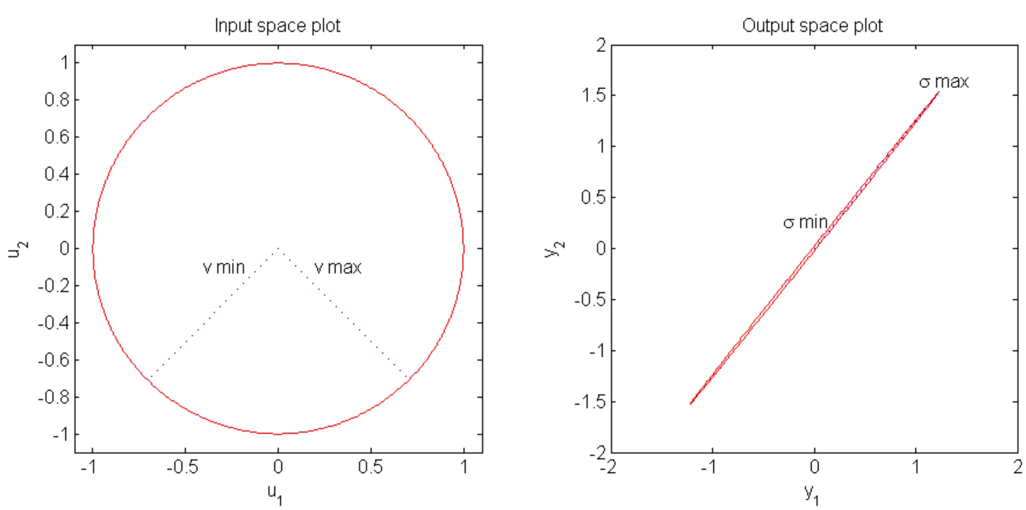

Figure 3.2: Input and output space of distillation column

From Figure 3.2 we can say, that it will be difficult to drive the system output in directions corresponding to σ_{min}, but it will be very easy to reach a high gain in directions corresponding to σ_{max}. The distillation column is a strongly ill-conditioned system as the major and minor axis proportion of the output space ellipse is very high .

The plot of output space in Figure 3.2 is quite illustrative, but as we are making an analysis of the ill-conditioned system behaviour, it is interesting to show the output response of the system in an open loop in the conventional form. Step response of both system outputs to signals in directions corresponding to highest and smallest condition number are displayed in Figure 3.3. The sizes of the input steps are comparable in both directions, but as it is obvious from the plots, the difference between the output response to a step in a strong and weak input direction is very large. The plots in Figure 3.3 should illustrate how difficult it may be to reach a desired output value in some directions. Sometimes the property that a system is not able to track a reference in some direction is not fundamental, but the fact that the system is not able to reject a disturbance may be of more interest. If a disturbance is introduced in a direction corresponding to a system small singular value, the control system may try to reject the disturbance by a large input value, but may not be able to do so because of the small gain in this direction. In an extreme case the input may rise without constraints. This may cause system input saturation or at least an undesired system consumption although the high input value has no significant effect. This is a reason for employing absolute value of input penalization to the MPC cost function.

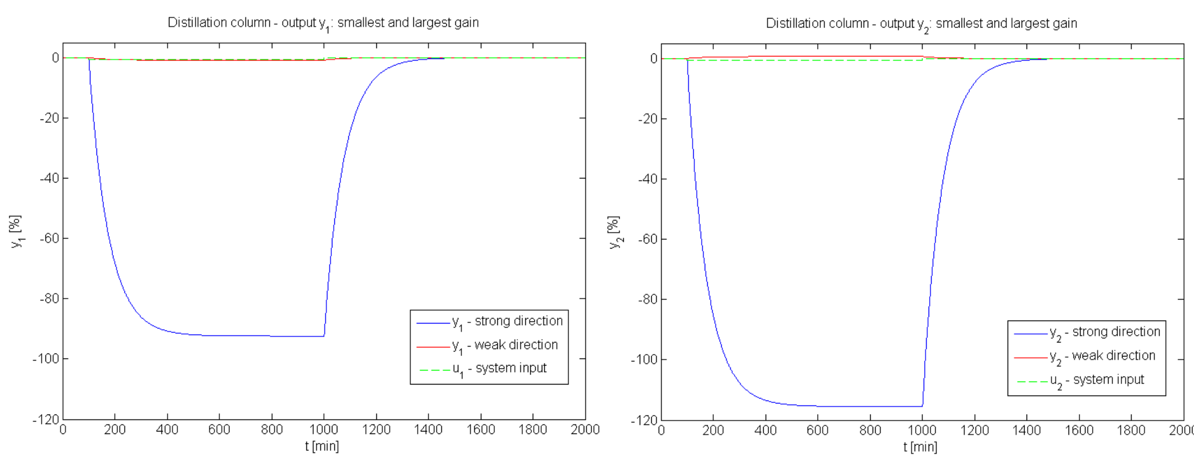

Figure 3.3: Open loop response of the distillation column to steps in strong and weak input directions

3.2.3 MPC design for distillation column

A Model Predictive Controller design and implementation for the distillation column will be described in this paragraph. To implement the proposed methods for ill-conditioned systems control some additional steps in the controller design procedure are necessary, as the controller is extended by the methods described in paragraphs 3.1.1 and 3.1.2. An offset free tracking MPC with Kalman filter will be employed. A state space model of an ill-conditioned system will be used for the design not only because the state space system representation is advantageous, as all the other system representations may be converted into it, but also because the Kalman filter will be used for the system state estimation. The general discrete system state space representation (2.35) is used. The ill-conditioned distillation plant has been discretized in Matlab using sampling period $T_s = 0.5$min. The system matrices for the distillation column are

$$\mathbf{A} = \begin{bmatrix} -0.0133 & 0 \\ 0 & -0.0133 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$\mathbf{C} = \begin{bmatrix} 0.878 & -0.8640 \\ 1.0820 & -1.0960 \end{bmatrix}, \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.15}$$

To introduce the offset-free tracking into MPC, the method described in section 2.3.4 will be used. A virtual disturbance acting on the system output will be introduced to the system, which will together with the Kalman filter ensure an offset free tracking of a reference signal. The augmented system with offset free tracking state space equations is described by equation (2.58), where the disturbance model matrices are $\mathbf{A}_d = \mathbf{I}$, $\mathbf{C}_d = \mathbf{I}$, which is a common setting in practical applications [2]. As it has been mentioned in paragraph 2.3.4, the output of MPC controller is a sequence of optimized input values for the whole prediction horizon. The frequency response analysis will be done for the system of distillation column to check the system robust stability. To perform standard system analysis, including frequency characteristics, it is useful to have a representation of the controller in terms of control law. We assume a prediction model of the system output (2.39) and a minimum energy quadratic cost function for reference tracking (2.45).

At this point it is necessary to implement the weak direction penalization described in paragraph 3.1.2. To make the equation (3.3) compact, the system state is extended by a system input and the cost function is written as

$$J\left(\mathbf{u}_k^{k+N}|\mathbf{x}_k,k\right) = \hat{\mathbf{e}}_k^T\mathbf{Q}_u\hat{\mathbf{e}}_k + \Delta\mathbf{u}_k^T\mathbf{R}\Delta\mathbf{u}_k \ , \tag{3.16}$$

where $\hat{\mathbf{e}}_k$ is the tracking error $\mathbf{e}_k$ extended by the system input $\mathbf{u}_k$, so that $\hat{\mathbf{e}}_k = \begin{bmatrix} \mathbf{e}_k \\ \mathbf{u}_k \end{bmatrix}$. The tuning of the criteria parameters remains the same as described in section 3.1.1, the weighting matrix $\mathbf{R}_u$ is now included in matrix $\mathbf{Q}$, resulting in a new matrix $\mathbf{Q}_u = \begin{bmatrix} \mathbf{Q} & 0 \\ 0 & \mathbf{R}_u \end{bmatrix}$. The second proposed method is implemented exactly as described in section 3.1.2. The resulting extended cost function with both methods implemented at the same time is

$$J\left(\mathbf{u}_k^{k+N}|\mathbf{x}_k,\mathbf{r}_k\right) = \hat{\mathbf{e}}_k^T\mathbf{Q}_u\hat{\mathbf{e}}_k + \Delta\mathbf{u}_k^T\mathbf{R}\Delta\mathbf{u}_k + \Delta\mathbf{u}_k^T\mathbf{H}_{wd}\Delta\mathbf{u}_k \ . \tag{3.17}$$

The control law can be obtained by optimization of equation 3.17. For MPC without constraints the optimal input trajectory $\mathbf{u}_{k+1}^\star$ can be found by solving a least squares problem. Using (2.39) the optimal control problem is

$$\begin{aligned} \min_{\mathbf{u}} J\left(\mathbf{u}_k^{k+N}|\mathbf{x}_k,k\right) = \left(\mathbf{r}_k - \bar{\mathbf{P}}\mathbf{x}_k - \bar{\mathbf{H}}\mathbf{u}_k\right)^T\mathbf{Q}_u\left(\mathbf{r}_k - \bar{\mathbf{P}}\mathbf{x}_k - \bar{\mathbf{H}}\mathbf{u}_k\right) + \\ + \Delta\mathbf{u}_k^T\mathbf{R}\Delta\mathbf{u}_k + \Delta\mathbf{u}_k^T\mathbf{H}_{wd}\Delta\mathbf{u}_k \ , \end{aligned} \tag{3.18}$$

with $\Delta\mathbf{u}_k = \mathbf{u}_k - \mathbf{u}_{k-1}$. Solution of this optimization problem is

$$\Delta\mathbf{u}_k^\star = (\bar{\mathbf{H}}^T\mathbf{Q}_u\bar{\mathbf{H}} + \mathbf{R} + \bar{\mathbf{H}}_{wd}^T\mathbf{R}_{wd}\bar{\mathbf{H}}_{wd})^{-1}(\bar{\mathbf{H}}^T\mathbf{Q}_u\bar{\mathbf{P}}) \ . \tag{3.19}$$

The optimized control sequence over the prediction horizon may be rewritten in terms of a control law as

$$\mathbf{u}_k^\star = -\mathbf{K}^x\mathbf{x}_k + \mathbf{K}^r\mathbf{r}_k + \mathbf{K}^u\mathbf{u}_{k-1} \ , \tag{3.20}$$

for a system with n_u inputs, n_y outputs and n_x states, the matrix $\mathbf{K}^x$ is composed of the first n_u rows and n_x columns of $\Delta\mathbf{u}_k^\star$, matrix $\mathbf{K}^u$ is composed of the following n_u rows and n_u columns of $\Delta\mathbf{u}_k^\star$ and finally matrix $\mathbf{K}^r$ is composed of the following n_u rows and n_y columns of $\Delta\mathbf{u}_k^\star$.To receive the representation of control input as described by equation (3.20), it is necessary to further augment the system representation described by equation (2.58). The system state $\mathbf{x}_k$

will be extended by a previous input value $\mathbf{u}_{k-1}$ and by reference $\mathbf{r}_k$. In order to receive the control sequence in the desired form, the output equation has to be expressed for tracking error $\mathbf{e}_k$ and regulator output $\mathbf{u}_k$. $\Delta\mathbf{u}$ will be used as a system input. The reasons for this system expansion is that the system needs to be represented in terms of variables present in equation (3.20). The resulting state space description with augmented system $\tilde{\mathbf{A}}$, $\tilde{\mathbf{B}}$, $\tilde{\mathbf{C}}$, $\tilde{\mathbf{D}}$ is then

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{u}_k \\ \mathbf{r}_{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_{k-1} \\ \mathbf{r}_k \end{bmatrix} + \begin{bmatrix} \mathbf{B} \\ \mathbf{I} \\ 0 \end{bmatrix} \Delta\mathbf{u}$$

$$\begin{bmatrix} \mathbf{e}_k \\ \mathbf{u}_k \end{bmatrix} = \begin{bmatrix} -\mathbf{C} & -\mathbf{D} & \mathbf{I} \\ 0 & \mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_{k-1} \\ \mathbf{r}_k \end{bmatrix} + \begin{bmatrix} \mathbf{D} \\ \mathbf{I} \end{bmatrix} \Delta\mathbf{u} \,. \tag{3.21}$$

Using these augmented system matrices the prediction matrices $\bar{\mathbf{P}}$ and $\bar{\mathbf{H}}$ used in the output equation (2.39) are composed according to equation (2.40).

To make the design procedure as universal as possible and to ensure the offset free tracking, a Kalman filter is used for the augmented system states estimation. As it is not in focus of this thesis, the Kalman filter design will not be described at this point. The Kalman filter for the distillation column has been designed using appropriate Matlab function with the process noise and disturbance covariance matrix $\mathbf{Q}_{kalman} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$ and the measurement noise covariance matrix $\mathbf{R}_{kalman} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$. With the Kalman filter introduced to the system, the controller design is almost finished. The weighting matrices $\mathbf{Q}_u$, $\mathbf{R}$ and the weighting coefficient α_{wd} of MPC have to be set so that the system performance is satisfying. Simulations for different settings of the weighting matrices will be shown. Not only the reference tracking of the resulting system will be studied, but also the frequency characteristics of different system configurations will be in our interest. MPC without the additional modifications for ill-conditioned systems is presented first, followed by simulations of the system with the two MPC modifications proposed in this thesis. The modifications will be first simulated one at a time to show the contribution of each of them. Both of them will be employed at the end. An effort will be made to design a controller with best possible performance and the robust stability conditions fulfilled. Robust stability will be presented for different uncertainty representations, namely the additive uncertainty, multiplicative input and output uncertainties and inverse additive uncertainty. These are the commonly used uncertainty representations, therefore, it will be useful to show the controller properties for them. It is not common to investigate robust stability for all the uncertainty representations, but it is done in this thesis to show the impact of MPC modifications when the uncertainties are represented in different parts of the system.

A system with MPC with modifications for an ill-conditioned system may have worse performance in terms of reference tracking comparing to conventional MPC, but as ill-conditioned systems are very sensitive to uncertanties, robust stability may be seen as an important factor when designing a controller. In order to analyze robust stability of different system configurations from section 2.4, it is necessary to rewrite the system equations in terms of appropriate inputs and outputs. To derive transfer functions for different uncertainty representations, the system block diagram has been modified in order to contain all the necessary inputs and outputs. This modified block diagram is displayed in Figure 3.4.

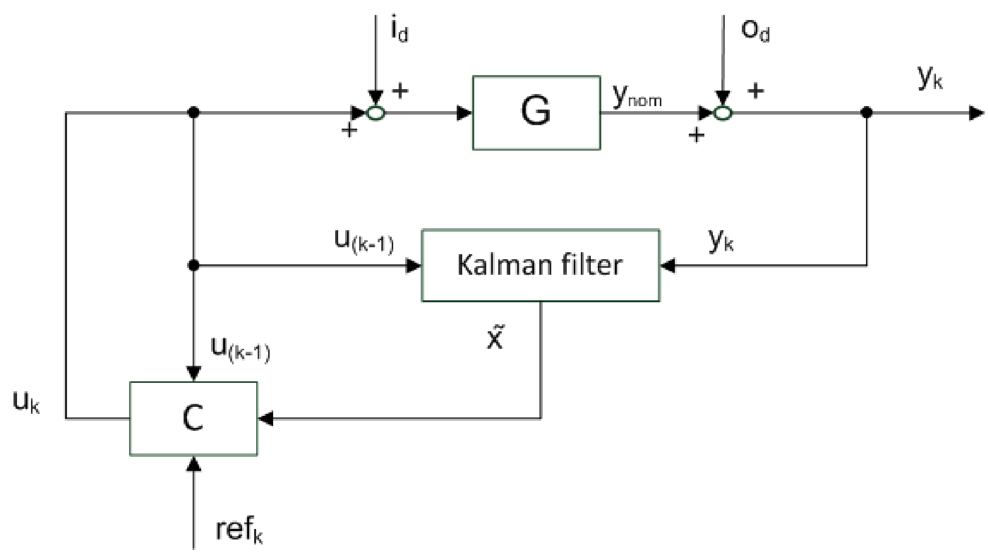

Figure 3.4: System block diagram for transfer functions derivation

Transfer functions will be expressed in terms of modified $M\Delta$ structure (section 2.4). This representation is suitable for stability analysis and makes the weighting matrices design possible.

Transfer functions for different uncertainty representations

Nomenclature from Figure 3.4 will be used for transfer functions derivations. According to Figure 2.5, the additive uncertainty is expressed as a transfer function from disturbance $\mathbf{o}_d$ to the system input $\mathbf{u}$. The Input multiplicative uncertainty is expressed as a transfer function from input disturbance $\mathbf{i}_d$ to system input $\mathbf{u}$, the output multiplicative uncertainty is expressed as a transfer function from the system output $\mathbf{y}$ to the nominal system output $\mathbf{y}_{nom}$, and finally the inverse additive uncertainty is expressed as a transfer function from system input $\mathbf{u}$ to the system output $\mathbf{y}$. These transfer functions were evaluated numerically in Matlab. For this evaluation, the augmented system state space description had to be rewritten in terms of the necessary inputs and outputs. The state space representation that has been used is

$$\begin{bmatrix} \mathbf{x}_{k+1} \\ \mathbf{u}_k \\ \mathbf{r}_{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{K}_u & \mathbf{B}\mathbf{K}_x \\ 0 & \mathbf{K}_u & \mathbf{K}_x \\ \mathbf{B}_{kfy}\mathbf{C} & (\mathbf{B}_{kfu}+\mathbf{B}_{kfy}\mathbf{D})\mathbf{K}_u & \mathbf{A}_{kf}+(\mathbf{B}_{kfu}+\mathbf{B}_{kfy}\mathbf{D})\mathbf{K}_x \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_{k-1} \\ \mathbf{r}_k \end{bmatrix} + \\ + \begin{bmatrix} & 0 & \mathbf{B} \\ \mathbf{B}_{fb}\mathbf{K}_r & 0 & 0 \\ & \mathbf{B}_{kfy} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{r}_k \\ \mathbf{o}_d \\ \mathbf{i}_d \end{bmatrix},$$

$$\begin{bmatrix} \mathbf{y}_k \\ \mathbf{u}_k \\ \mathbf{e}_k \\ \mathbf{y}_{nom_k} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & \mathbf{D}\mathbf{K}_u & \mathbf{D}\mathbf{K}_x \\ 0 & \mathbf{K}_u & \mathbf{K}_x \\ -\mathbf{C} & -\mathbf{D}\mathbf{K}_u & -\mathbf{D}\mathbf{K}_x \\ \mathbf{C} & \mathbf{D}\mathbf{K}_u & \mathbf{D}\mathbf{K}_x \end{bmatrix} \begin{bmatrix} \mathbf{x}_k \\ \mathbf{u}_{k-1} \\ \mathbf{r}_k \end{bmatrix} + \begin{bmatrix} \mathbf{D}\mathbf{K}_r & \mathbf{I} & \mathbf{D} \\ \mathbf{K}_r & 0 & 0 \\ \mathbf{I}-\mathbf{D}\mathbf{K}_r & -\mathbf{I} & -\mathbf{D} \\ \mathbf{D}\mathbf{K}_r & 0 & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{r}_k \\ \mathbf{o}_d \\ \mathbf{i}_d \end{bmatrix}. \tag{3.22}$$

From the state representation in equation (3.22) it is possible to evaluate all the above mentioned transfer functions. These will be used in order to get the frequency responses of the uncertainty models. Setting the weighting matrices $\mathbf{W}$ for different uncertainty representation it will be possible to decide on robust stability of the system. Using these transfer functions, the frequency response for the different uncertainty representations will be plotted. The nominal system model will be used to create the weighting function for the different uncertainty representations according to the derived relations described by equations (2.9) - (2.76).

The frequency characteristics of the different augmented $M\Delta$ structures are influenced by the parameters of augmented MPC for ill-conditioned plants. The influence of the individual MPC modifications described in sections 3.1.1 and 3.1.2 will be shown on simulations in section 3.2.4. At this point it is important to note, that by a right choice of the weighting matrices of the augmented MPC it is always possible to improve the stability in comparison with the conventional MPC without modifications. Robust stability may be achieved for many systems, but as will be shown in simulations, achieving robust stability is not always assured.

3.2.4 Simulations

In this paragraph both quality of control and robust performance of the distillation column will be presented. System with conventional MPC without the additional modifications for ill-conditioned plants will be presented first. The following simulations will be compared with this MPC in terms of reference tracking and robust stability. The two presented modifications of MPC will be applied one at a time to show their contribution. As these methods are easily applicable at the same time, a controller that takes advantage of both proposed modifications will be presented at the end.

Conventional MPC

MPC controller has been designed according to the steps described in section 3.2. The weights added to the cost function in order to improve the controller behaviour for ill-conditioned systems were set to zero. The other tuning parameters of the controller were set as follows:
MPC horizon length $N_{horizon} = 30$, weighting matrices of the cost function were set as

$$\mathbf{Q}_u = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \alpha_{wd} = 0\,, \tag{3.23}$$

and covariance matrices were set as

$$\mathbf{Q}_{Kalman} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}, \mathbf{R}_{Kalman} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}. \tag{3.24}$$

A reference signal with step changes in both weak and strong directions of the system has been chosen to show the properties of the system with MPC. To show that the system is strongly interactive, the steps were first done in individual channels, and later in both input channels at the same time. The reference tracking is displayed in the first two plots of Figure 3.5.

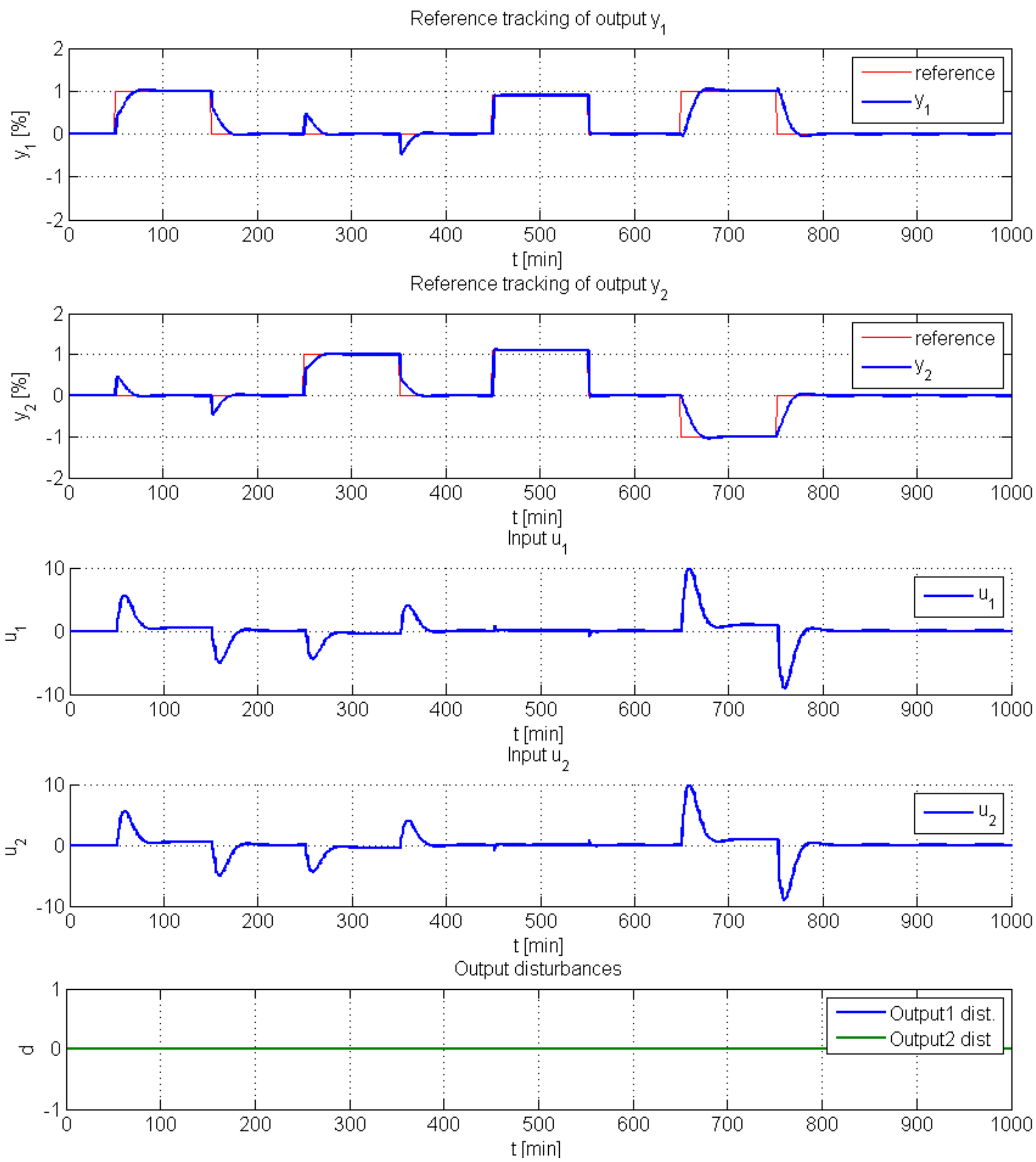

Figure 3.5: Simulation of system with MPC without modifications

First there has been a step change done in the reference for the first output at time $t = 50$min. The first output of the system is able to track the reference, but as the system is strongly interactive, there has to be an action taken by input u_2 in order for the output y_2 to track the zero reference signal. A similar test is done for the second output y_2 at time $t = 250$min, the system is able to track the reference signal, and similarly to the previous case, an action has to be taken by input u_1 in order to track the zero reference by output y_1. At time $t = 450$min a step in the direction corresponding to the system largest singular value $\overline{\sigma}$ is made, followed by a step change in the direction corresponding to the system smallest singular value $\underline{\sigma}$ at time $t = 650$min. Both system outputs track the reference signal. In the case of step in directions corresponding to the largest singular value, the response is much faster, and as it can be seen from the third and fourth plot of Figure 3.5, the necessary input action is much smaller than in the case of step in directions corresponding to the system smallest singular value. From the input signals plot in Figure 3.5 it is obvious that change of inputs in opposite directions has a larger impact on the system output change than a change of the inputs in opposite directions. This confirms the results of system analysis in section 3.2.2.

At this point it is interesting to show how the system will be able to track reference signals when a disturbance is introduced. An output disturbance in different directions will be applied. Similarly to the disturbance-free reference tracking, both the worst and best directions of the disturbance signal will be introduced. These directions are decided based on a disturbance condition number $\gamma_d(\mathbf{G}_{LV})$ (section 2.2.3). For the simulations a disturbance with directions $\mathbf{d}_1 = \begin{bmatrix} 0.9 \\ 1.1 \end{bmatrix}$ with $\gamma_d(\mathbf{G}_{LV}) = 1.87$ and $\mathbf{d}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with $\gamma_d(\mathbf{G}_{LV}) = 140.9$ were chosen. According to the $\gamma_d(\mathbf{G}_{LV})$ value the disturbance $\mathbf{d}_1$ is easy to reject. As the disturbance $\mathbf{d}_2$ value of $\gamma_d(\mathbf{G}_{LV}) = 140.9$, is very close to the system condition number $\gamma(\mathbf{G}_{LV}) = 141.7$, it is expected to be very difficult to reject.

The simulation is shown in Figure 3.6. Both outputs are tracking a reference starting from time $t = 50$min, first disturbance $\mathbf{d}_1$ is introduced at $t = 150$min. As it is expected for a disturbance with direction close to the strong output direction, the disturbance is easily rejected with small input actions necessary. The disturbance in direction $\mathbf{d}_2$ is applied at time $t = 350$min. When compared to the disturbance $\mathbf{d}_1$, the disturbance in this direction needs a higher input action in order to be rejected and the rejection takes longer in this case. At this point it is important to emphasize the size of input actions necessary to reject the disturbance. These are from region $u_{1,2} \in \langle -10;\ 2\rangle$, which are undesirably high values. These input actions should lie within a region $u_{1,2} \in \langle -1;\ 1 \rangle$.

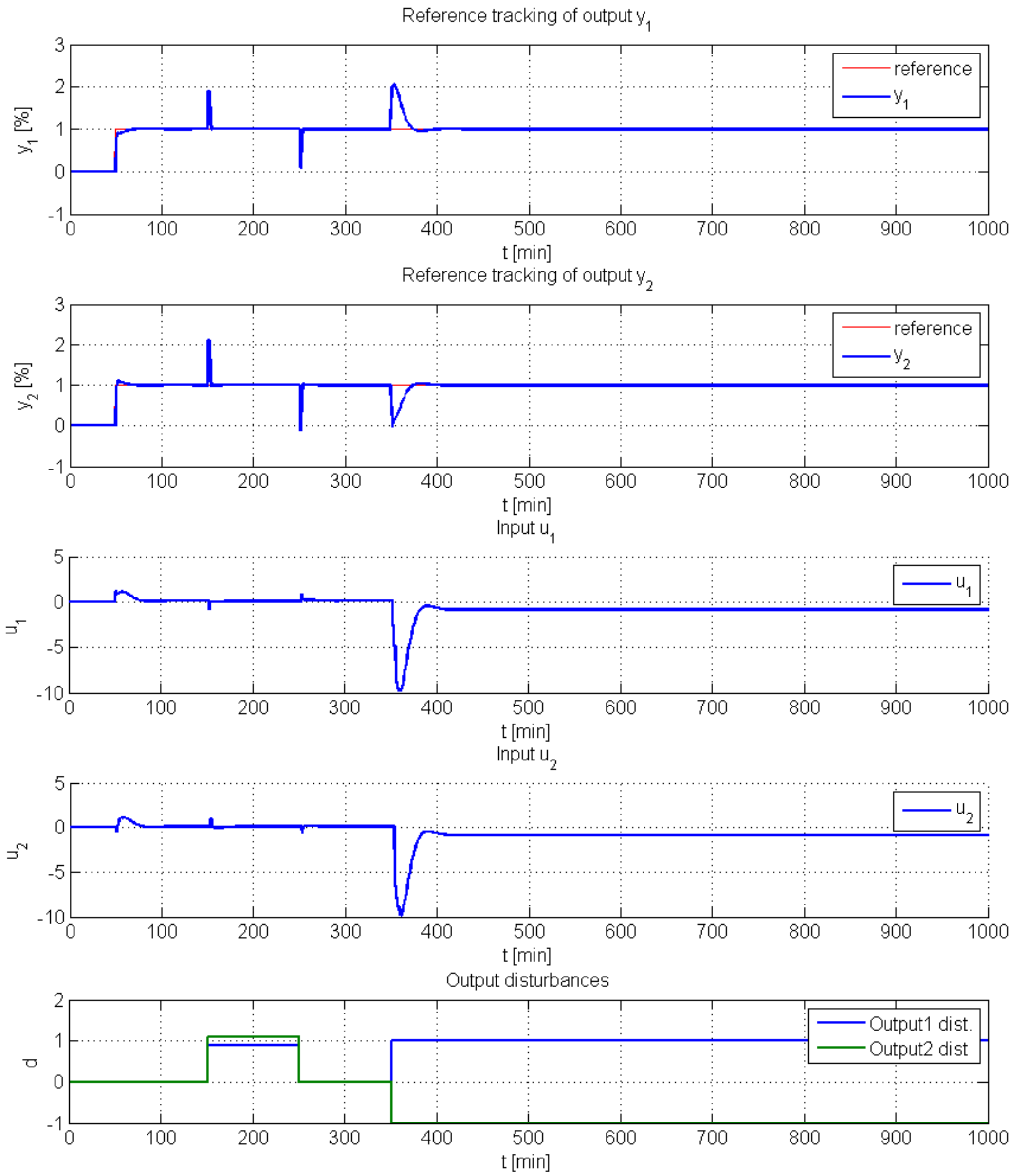

Figure 3.6: Disturbance rejection of system with MPC without modifications

As the MPC without constraints is applied to the system, the input actions may rise towards undesired values and even towards infinity. It is not the case of the presented ill-conditioned distillation column, but for some combinations of disturbances and reference signals, this could be the case for many ill-conditioned systems.

Robustness analysis for MPC without modifications

To approve robust stability, the frequency characteristics of the different uncertainty representations were plotted for the distillation column with MPC. In this thesis the aim is not to make a perfect model of distillation column disturbance and represent it in configurations of the system. The aim is to show how the MPC modifications influence the frequency characteristics of the system. Four different uncertainty representations were chosen, but these represent different uncertainties of a system. The weighting functions were calculated based on the derivations in section 2.4.1, a 10% uncertainty for all chosen uncertainty types is considered. This corresponds to $\varepsilon = 0.1$ in equations in section 2.4.1. Frequency responses for the four different system configurations are displayed in Figure 3.7.

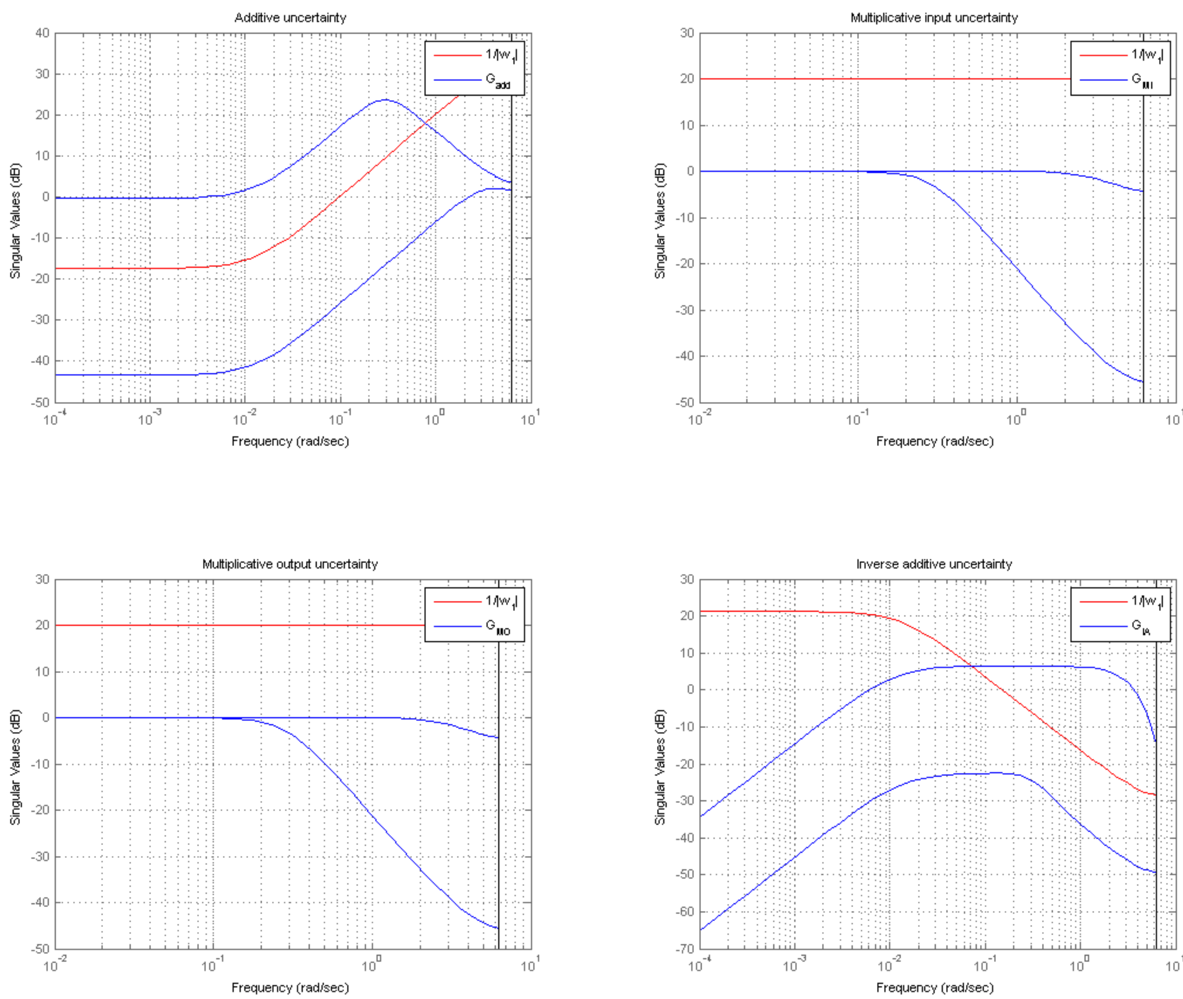

Figure 3.7: Frequency characteristics of different uncertainty representations

As it can be seen from the Figure 3.7, distillation column with MPC is robustly stable for the input and output multiplicative uncertainty representation. On the other hand side, the system with MPC is not robustly stable for the additive uncertainty and the inverse additive uncertainty. The frequency characteristics of the distillation column with the augmented MPC will be compared to the frequency characteristics in Figure 3.7. By appropriate settings of the weighting matrices a robustly stable system will be a desired result of our effort.

MPC with input absolute value penalization

The same reference signal used for simulations in Figure 3.5 will be used for simulations of all the systems with modified MPC. This reference signal shows the reference tracking abilities of different MPC modifications and enables us to compare their behaviour.
The simulation for the first modification of MPC with input absolute value penalization, with the weighting matrix with input penalization

$$Q_u = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 13.2 & 0 \\ 0 & 0 & 0 & 13.2 \end{bmatrix} \tag{3.25}$$

is displayed in Figure 3.8. All the other MPC tuning parameters were set to the same values as in equations (3.23) and (3.24).

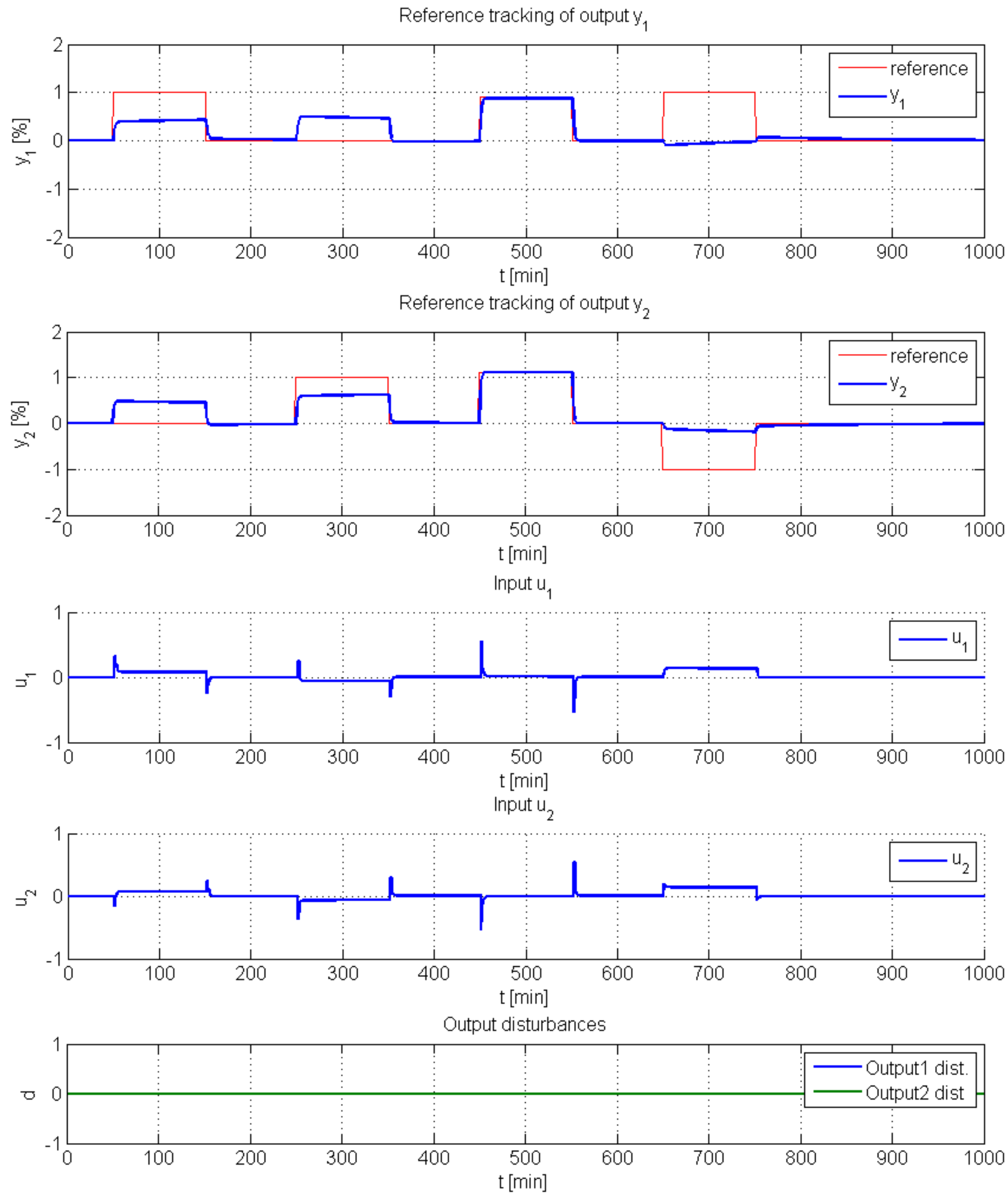

Figure 3.8: Simulation results for MPC with input absolute value penalization

From the first two plots of Figure 3.8 it is possible to see, that the controller is not able to track a reference when a change of only one output is desired. This may be seen from reference changes at time $t = 50$min for change in output y_1 and at time $t = 250$min for change in output y_2. The reference signal is tracked with a steady state offset if a change corresponding to the direction of the largest system singular value $\bar{\sigma}$ is introduced to the system. This occurs at time $t = 450$min. A reference in a direction corresponding to the smallest system singular value $\underline{\sigma}$

can not be tracked, as it may be seen at time $t = 650$min.

The disturbance rejection for the same disturbances as in section 3.2.4 is presented in Figure 3.9.

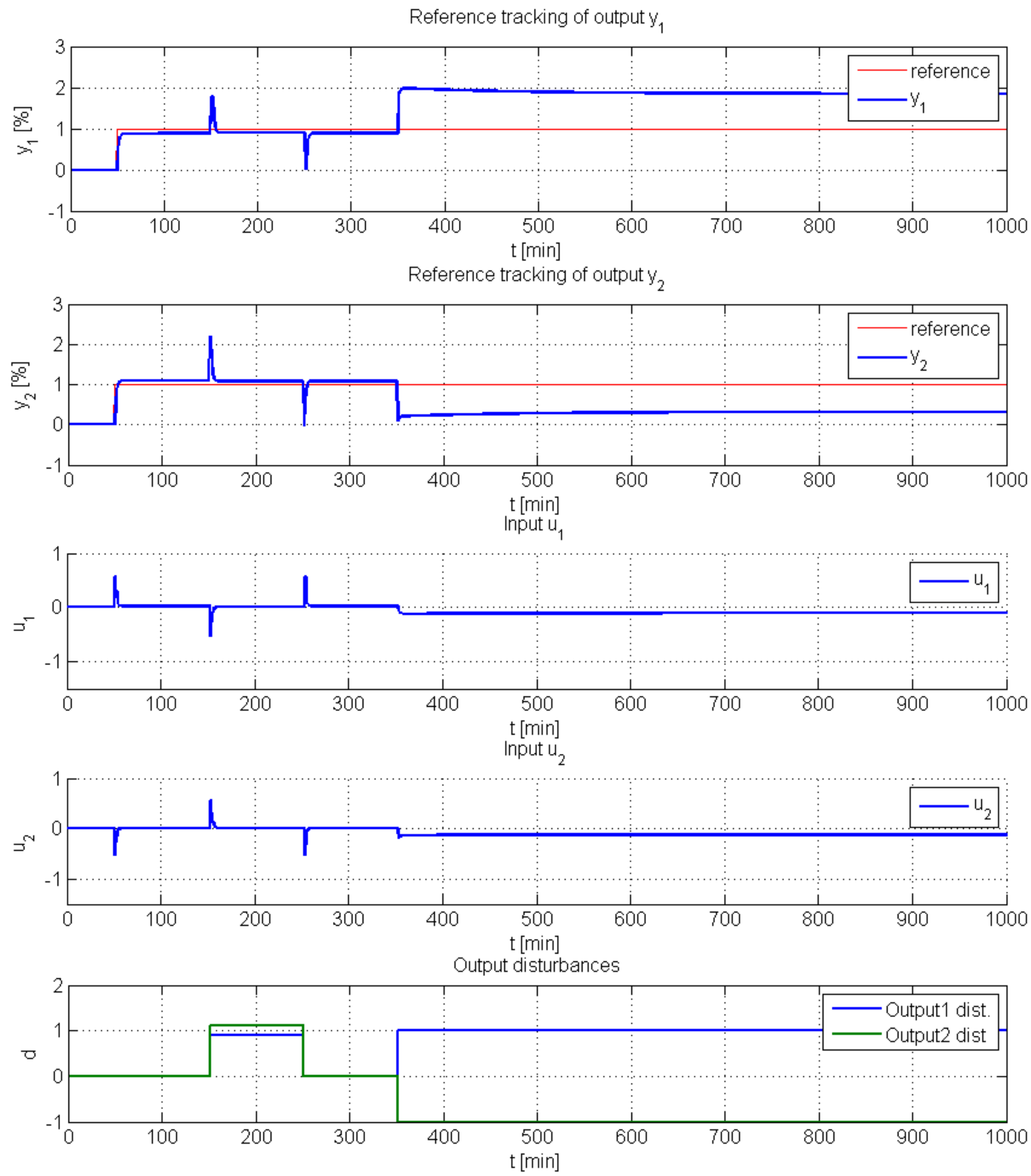

Figure 3.9: Disturbance rejection of system with MPC with input absolute value penalization

A steady state offset of reference tracking and disturbance rejection is a drawback of this method. A reference is tracked if a disturbance is introduced in a direction close to the system strongest output direction $\bar{\mathbf{u}}_y$, which is introduced to the system at time $t = 150$min in Figure

3.9. A disturbance in a direction close to the weak output direction $\underline{\mathbf{u}}_y$ can not be rejected by the system as it may be seen at $t = 350$min in the first two plots of Figure 3.9.The reference tracking abilities and disturbance rejection of this MPC may seem unsatisfying, but the advantage of this implementation is the achieved robust stability for the additive uncertainty model. The size of input action should be emphasized at this point. It does not cross the desired output region $u_{1,2} \in \langle -1; 1 \rangle$. As the robust stability conditions for the multiplicative input and multiplicative output uncertainty representations have been fulfilled even for the conventional MPC in Figure 3.7 and are fulfilled for all the following settings, these frequency characteristics will not be displayed. The frequency characteristics for additive uncertainty model are plotted in Figure 3.10. The robust stability is achieved for the MPC weighting matrix $\mathbf{Q}_u$ set according to equation (3.25).

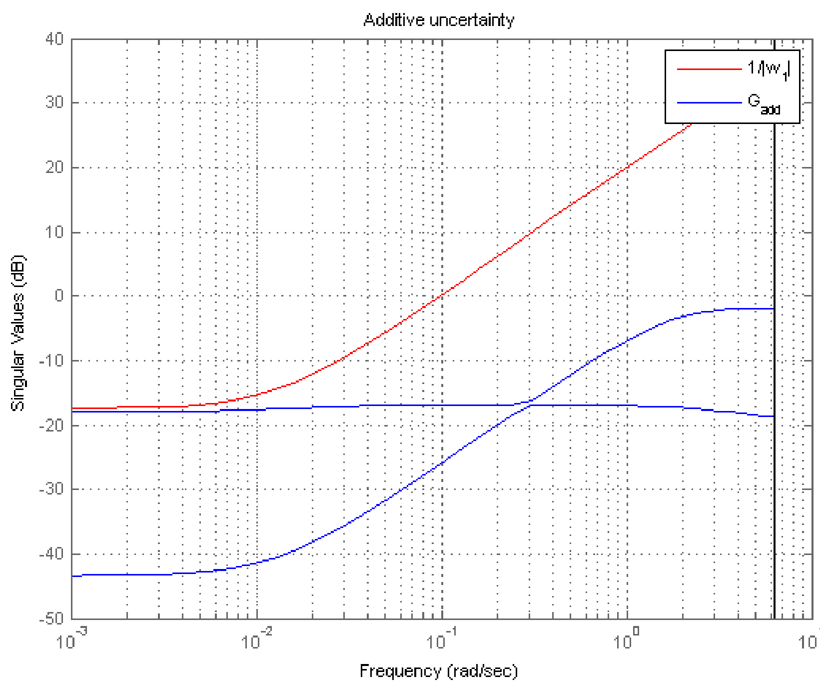

Figure 3.10: Frequency characteristics of additive uncertainty model for MPC with absolute value of u penalization

Robust stability for inverse additive uncertainty representation is not achieved for any setting of weighting matrix $\mathbf{Q}_u$. The frequency characteristics are displayed in Figure 3.11 for the same settings as in Figure 3.10. To compare the results with the MPC without modifications, the frequency characteristics of this controller without modifications are plotted in green in Figure 3.11. As it may be seen from the Figure, the system is stable assuming the 10% uncertainty at low frequencies, but the stability is violated at higher frequencies.

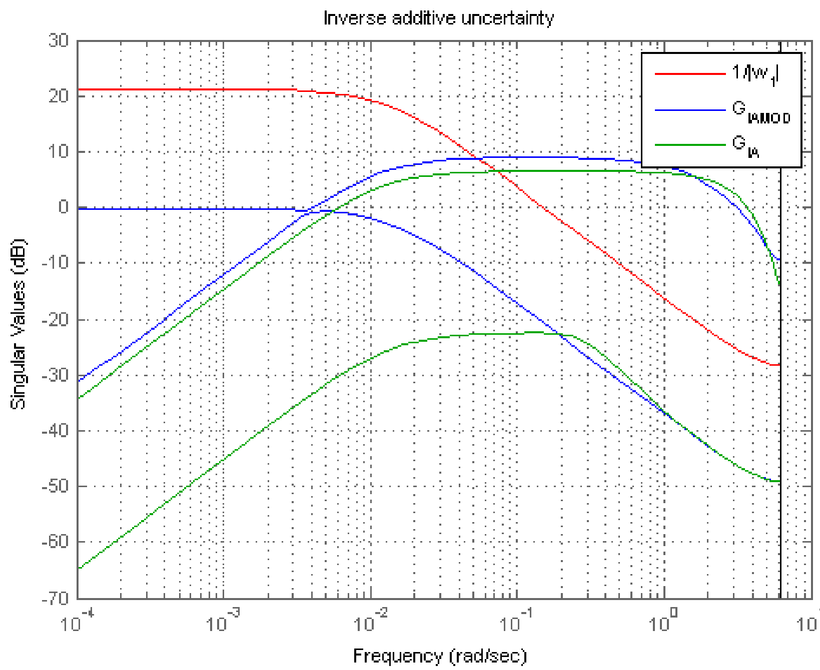

Figure 3.11: Frequency characteristics of the inverse additive uncertainty representation for MPC with input absolute value penalization

MPC with movement in weak input directions penalization

The simulation results for an MPC with movement of input in weak directions penalization is presented in Figure 3.12. The same reference as in previous cases has been used to show the system behaviour. The MPC settings remain unchanged from previous simulations except that the penalization of input absolute value is canceled and the movement of the system input is penalized by a weighting coefficient $\alpha_{wd} = 5600$. This value delivers best results in terms of system stability, which will be described in the next paragraph.

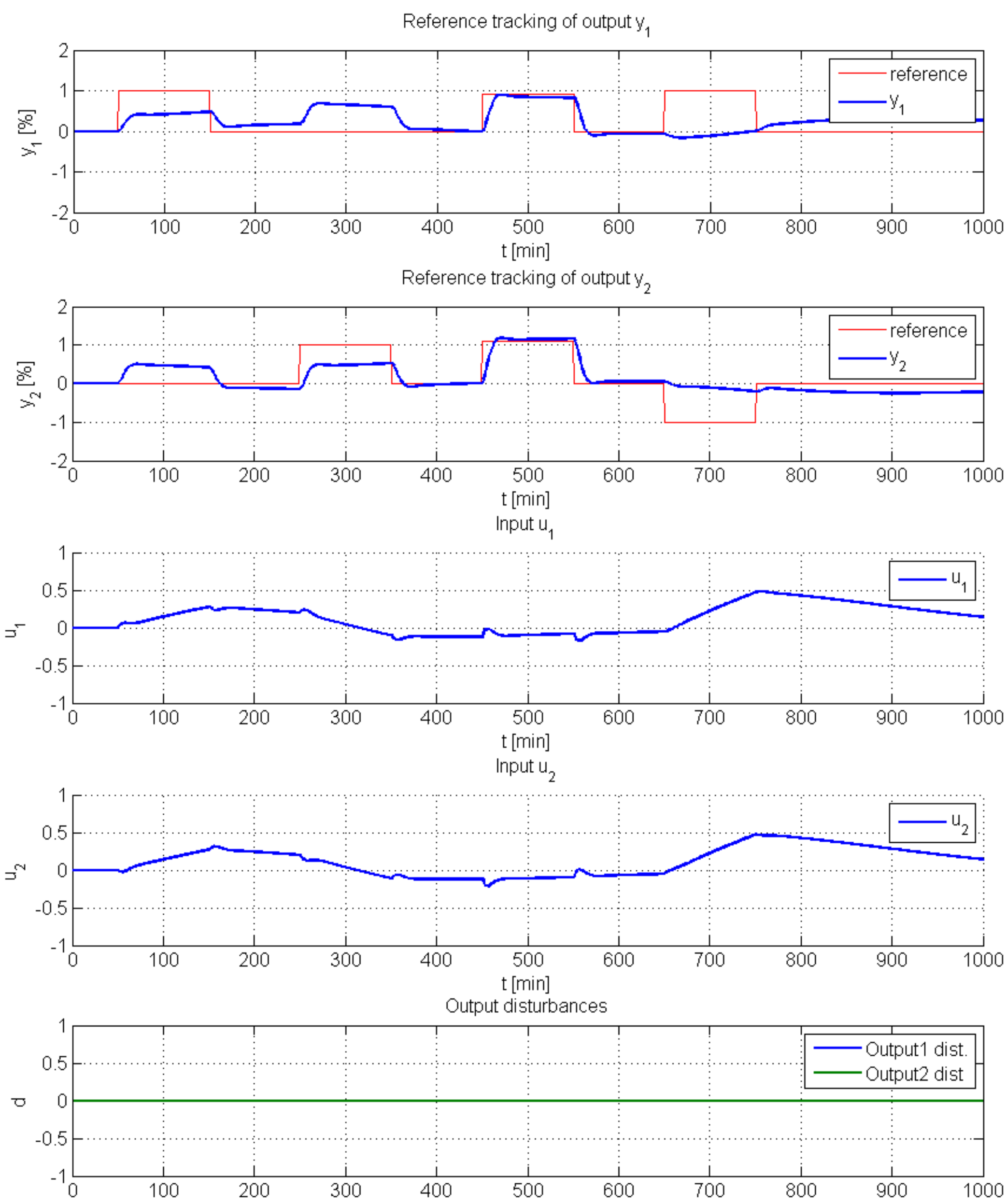

Figure 3.12: Simulation results for MPC with increment in weak input directions penalization

As it may be seen from the first two plots of Figure 3.12, the outputs are not able to track reference if a change of reference is done in only one of the outputs. This is a case of reference changes at times $t = 50$min and $t = 250$min. The reference is tracked if a change is made in a direction close to the output direction corresponding to the largest system singular value $\overline{\sigma}$. This change occurs at time $t = 450$min. The system is not able to track a reference which requires a change in output in a direction which is close to the direction corresponding to the smallest system singular value $\underline{\sigma}$. This change occurs at time $t = 650$min. Due to the movement in weak input directions penalization the input actions of the system are not large, as the system is not trying to push a large input values in directions where it has no significant effect. The reference tracking in the directions corresponding to the largest system singular value is offset free, but the settling time is large, as it may be seen from the step change at $t = 450$min.

The disturbance rejection of system with MPC extended by movement in weak input directions penalization are plotted in Figure 3.13. The reference is tracked with some error, which slowly converges to zero. A disturbance in direction $\mathbf{d}_1$ is easily rejected, as its direction is close to the strong system output direction. This may be seen at time $t = 250$min. A disturbance in direction $\mathbf{d}_2$, which has a direction close to the weak system output direction $\underline{\mathbf{u}_y}$ is difficult to reject. This disturbance is introduced to the system at time $t = 350$min. The disturbance is rejected very slowly, with the input vector values to $\mathbf{u} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, which is a lower limit for input value.

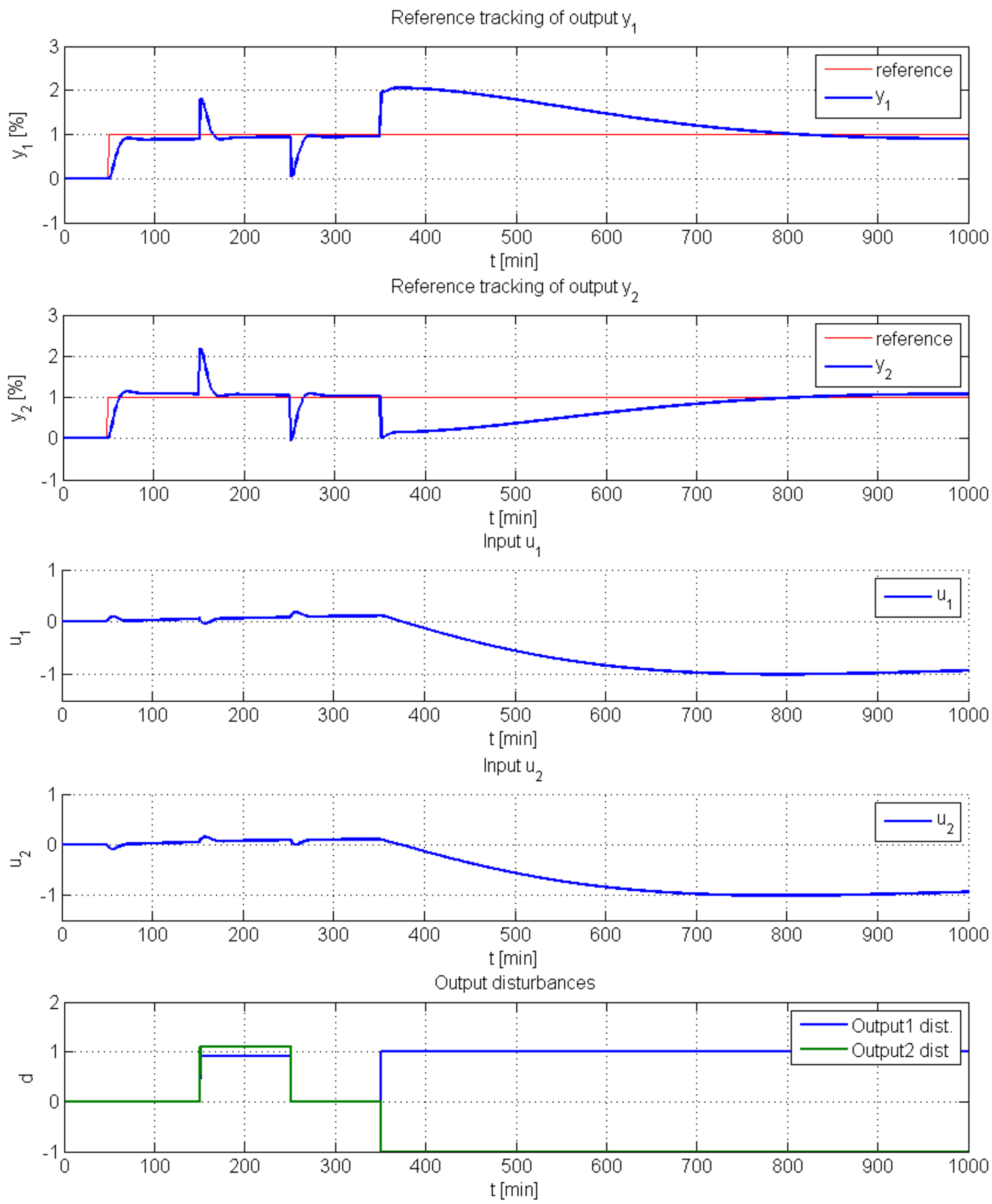

Figure 3.13: Disturbance rejection of system with MPC with increment in weak input directions penalization

Robust stability for additive uncertainty model is not achieved by any settings of weighting coeficient α_{wd} of this criteria. The frequency characteristics for $\alpha_{wd} = 5600$ are shown in Figure 3.14. The frequency characteristics for MPC without modifications are plotted in green to show that the blue characteristics of the MPC with movement in weak input direction penalization improves the stability of the system, but it is not possible to push the largest singular value $\bar{\sigma}$

frequency characteristics below the weighting function $\frac{1}{W_1}$.

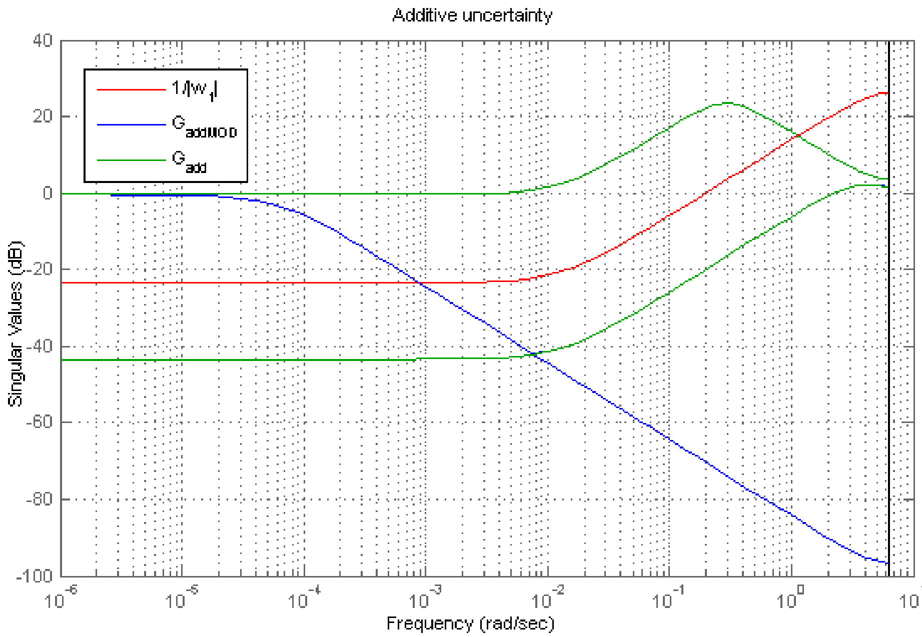

Figure 3.14: Additive uncertainty model frequency characteristics for MPC with increment in weak input directions penalization

For the Inverse additive uncertainty representation the robust stability is not achieved either. Similarly to the previous plot, both conventional MPC and MPC with increment in weak input direction penalization are presented in Figure 3.15. As it may be seen from the frequency characteristics in this figure, it is not possible to push the frequency characteristics below the weighting function $\frac{1}{W_{Inv}}$.

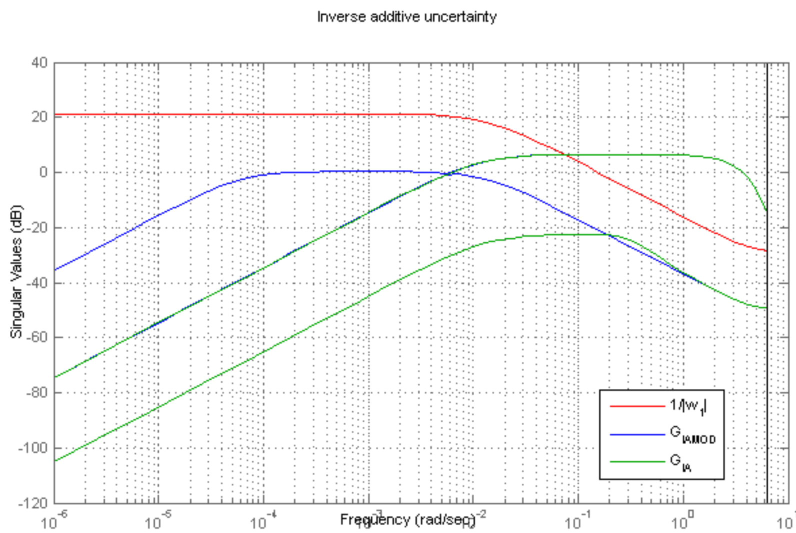

Figure 3.15: Inverse additive uncertainty model frequency characteristics for MPC with movement in weak input directions penalization

MPC with absolute value of input and movement in weak input directions penalization

Finally MPC with both modifications has been designed. The best results in terms of robust stability were achieved for the weighting coefficients set as:

$$\mathbf{Q}_u = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 5.8 & 0 \\ 0 & 0 & 0 & 5.8 \end{bmatrix} \mathbf{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \alpha_{wd} = 1200 \ .$$

From the first two plots of Figure 3.16 it is possible to see, that the system is able to track a reference when it changes so that a step corresponding to a strong system output direction is required ($t = 450$min). The system is not able to track a reference if a change is done only in one of the inputs at a time ($t = 50$min and $t = 250$min) or in a direction which requires an input action in a weak input direction $t = 650$min.

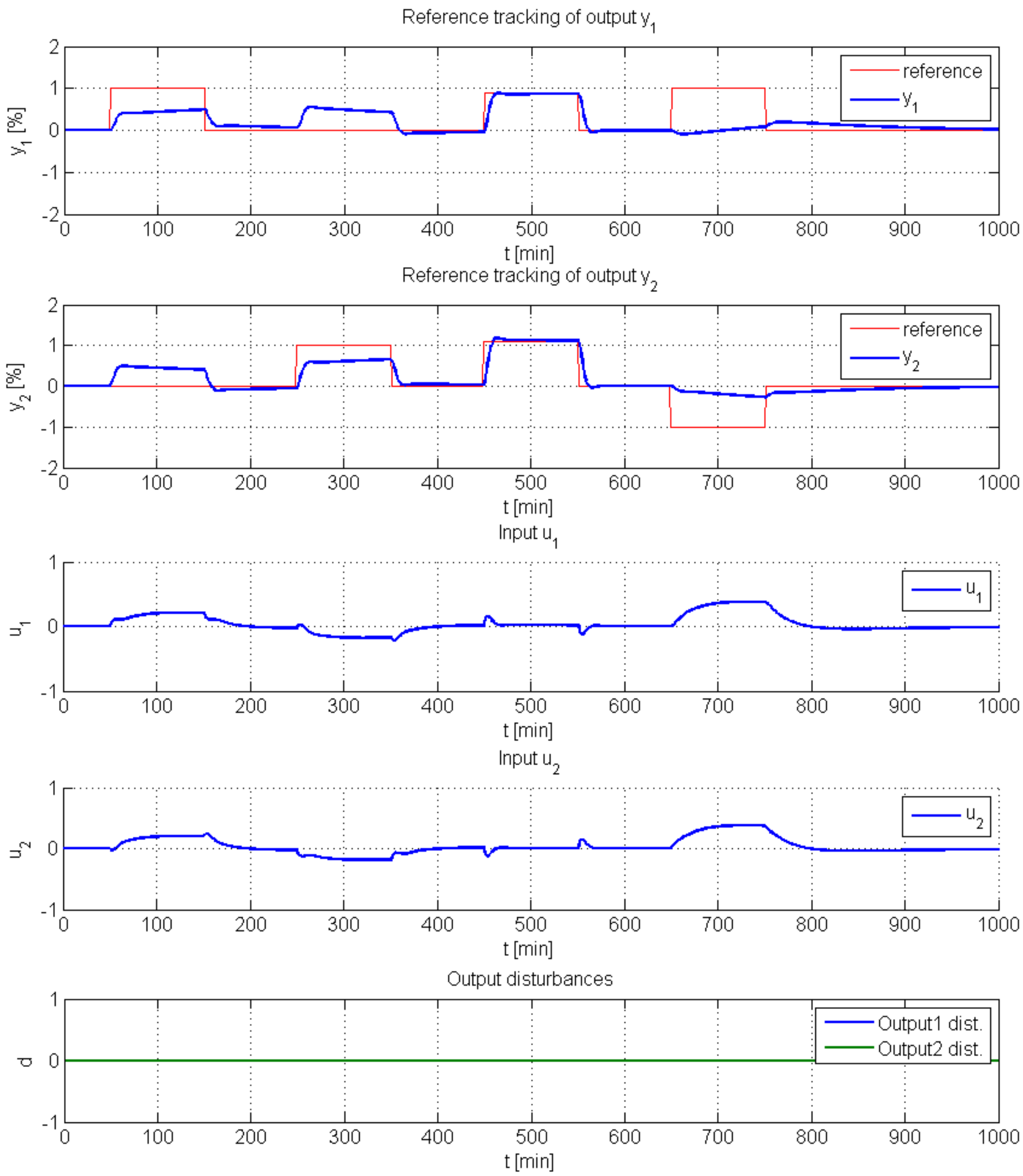

Figure 3.16: Simulation results for MPC with input absolute value and movement in weak input directions penalization.

The disturbance rejection abilities of the system are examined in Figure 3.17. The system tracks a reference signal with a small steady state offset, which is caused by input absolute value penalization introduced to the system. A disturbance in direction $\mathbf{d}_1$ ($t = 150$min) is rejected without a need for a large input action. A disturbance in a direction $\mathbf{d}_2$, which is close to the system weak output direction, is not rejected ($t = 350$min). At this point it is important to note that the input actions are within the allowed region $u_{1,2} \in \langle -1;\ 1 \rangle$.

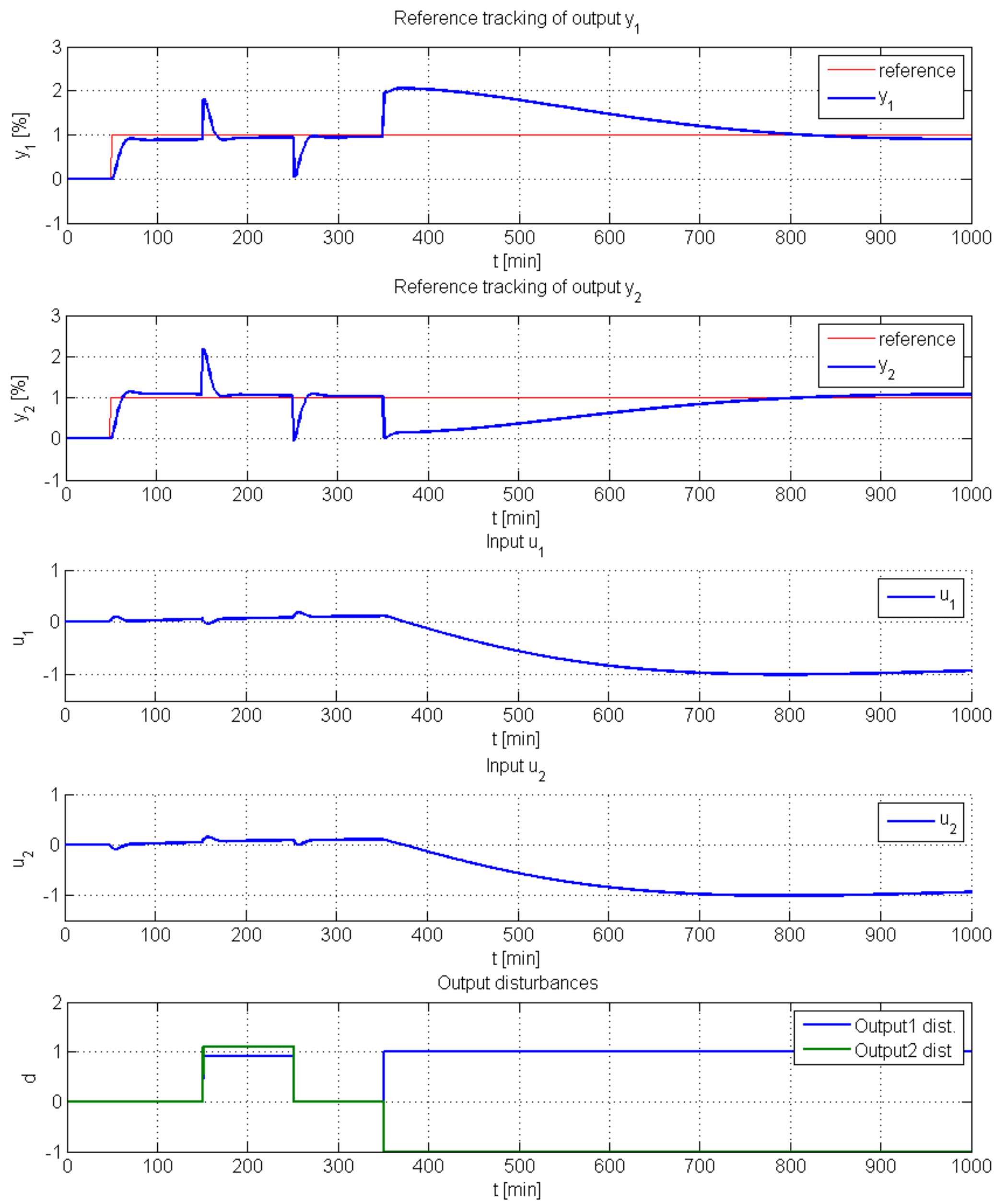

Figure 3.17: Disturbance rejection of system with MPC with increment in weak input directions and input absolute value penalization

Robust stability has been achieved for additive uncertainty model. The frequency characteristics for this model are plotted in Figure 3.18 together with the conventional MPC characteristics to emphasize the stability. The robust stability is achieved for input and output multiplicative model as well. The frequency characteristics for these representations are not displayed as the robust stability conditions for these representations were satisfied even for the MPC without any modifications.

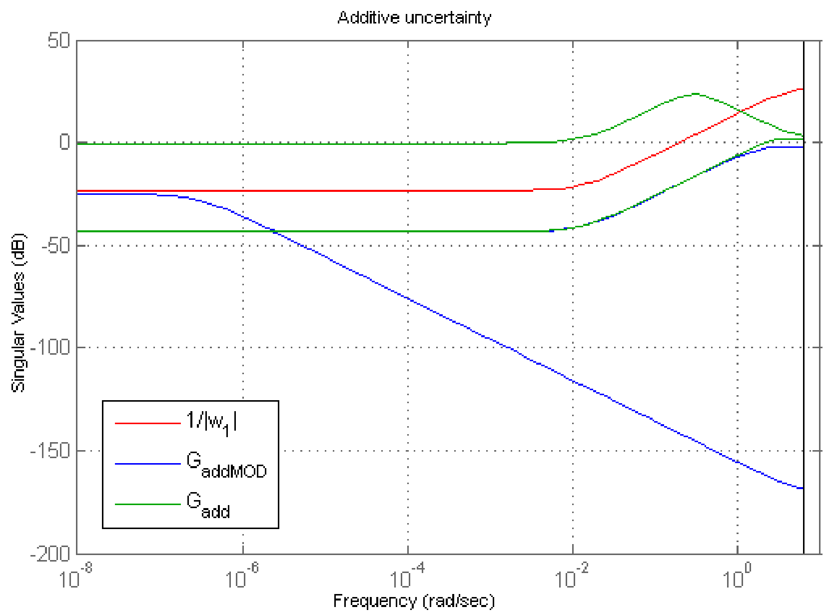

Figure 3.18: Frequency characteristics for additive uncertainty representation for MPC with both modifications

Robust stability for a system with the inverse additive uncertainty representation has not been achieved by any regulator parameters settings. The frequency characteristics for this uncertainty representation are displayed in Figure 3.19 together with the conventional MPC frequency characteristics (green line). It is important to point out that the stability has not been made worse by a huge difference. Although there is a small deterioration in the stability, it is not a remarkable difference.

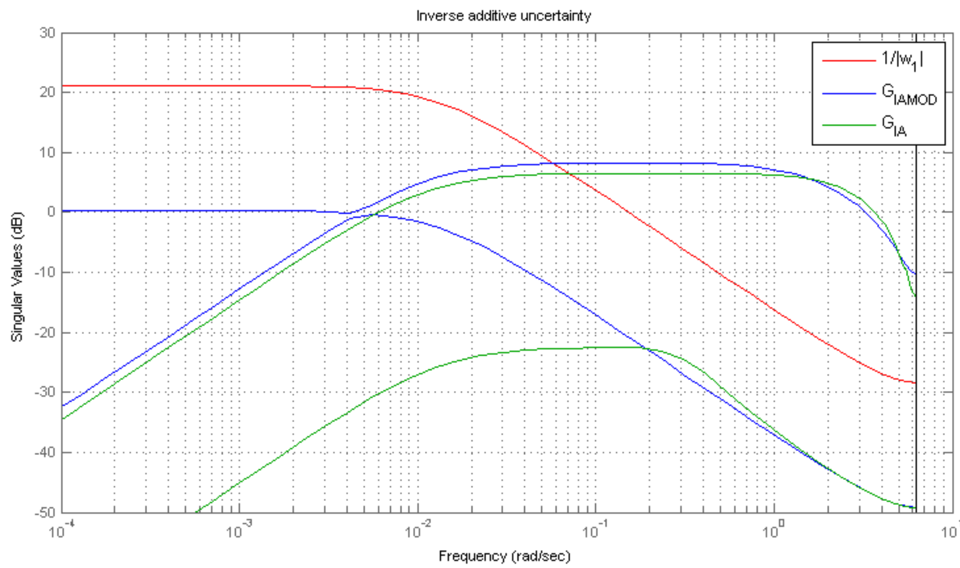

Figure 3.19: Frequency characteristics for inverse additive uncertainty representation for MPC with both modifications

Chapter 4

Conclusion

The main goal of this thesis was to modify MPC controller in order to be able to control ill-conditioned systems. First an overview of methods that may be used as measures of MIMO system conditioning and MIMO system interactions was done. This overview has been based on a background research on these topics. The up-to-date Gramian based interaction measures were presented in this thesis in addition to the widely-used conventional ill-conditioning and interaction measures, which are the condition number and RGA. The second part of the background research was concerned with MPC applications for ill-conditioned systems. The background research is presented in Appendix A of this thesis.

Two extensions of Model Predictive Controller were proposed in this thesis, based on the acquired knowledge in the background research. The first proposed method extends MPC cost function by a term that penalizes absolute value of the system input. This adjustment is necessary to prevent undesirably high system inputs, which may be introduced by a controller when trying to achieve a reference in a weak system input direction.

The second method extends MPC cost function by another member that penalizes movement of the system input in weak input directions. This method uses deeper knowledge about the system gained by system Singular Value Decomposition. Increment of the system inputs in the input directions corresponding to system small singular values is penalized. In order to do so, an additional term is introduced to the MPC cost function. This is advantageous as it directly penalizes the undesired controller movement, as the movement of the controller in the weak input directions has a small impact on the reference tracking abilities of the regulator.

To show the gained results on a practical example, a model of a distillation column has been used. The system properties are analysed in section 3.2.2. The distillation column shows to be strongly ill-conditioned with significant interactions. Reference tracking and disturbance rejection abilities are examined in a series of simulations in order to show system behaviour. To emphasize the contribution of the two proposed methods, simulations for MPC without these modifications are performed first. The results of these simulations are compared with results for the individual methods of MPC modifications for ill-conditioned systems. Both methods are combined at the end to show that advantage can be taken of using these two methods at the same time.

The performed simulations have shown that reference tracking is not improved by extending the MPC by the proposed methods. On the other hand, the proposed methods limit the system input values in order to lie within the acceptable region. Moreover an improvement of stability has been proved in section 3.2.4. Different configurations of system with uncertainties have been used to show the impact of the modifications on the frequency characteristics of transfer functions for different uncertainty representations. The method with MPC input absolute value penalization pushes the frequency characteristics of a system down at lower frequencies, and on the other hand, the penalization of MPC input increment in weak input directions penalization helps to bring down the frequency characteristics at high frequencies. The combination of these two methods is therefore useful when stability needs to be improved or when robust stability is to be achieved.

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Appendix A

Background research

1. On a new measure of Interaction for Multivariable Process Control [5] (1966), Correspondence

Edgar Bristol

An interaction measure for multivariable system control is proposed in this article. The author assumes a linearized, time- invariant, multivariable process, described by a square gain matrix. The gain matrix by itself is a poor tool for classifying control properties of a system, as it is strongly dependent on scaling and choice of units. Scaling becomes confusing especially with large systems with many inputs and outputs. All the conventional descriptions of matrix properties, as norms, determinant and eigenvalues, are scaling dependent, therefore unsuitable as a measures of process structure. An interaction measure describes the transfer function between a given manipulated variable and a given controlled variable, which is affected by the complete control of all other controllers in the system. The article proposes a ratio between two gains as an interaction measure. The first gain (ϕ_{ij}) is the process gain in an isolated loop, all other loops are not closed. The second gain (ϕ_{ji}) is the gain in the same loop when all other loops in the system are closed. The ratio of these two gains defines an array with elements $\mu_{ij} \triangleq \phi_{ij}\phi_{ji}^{-1}$. This relation is called a condition number. The important properties of this measure are presented. The most important of them are the facts that the measure is invariant under scaling, delivers a reasonable indicator of nearly singular gain matrices, and that it shows up in calculation of changes introduced in a control system, caused by changes in process parameters. There are a few examples of use of this measure of interaction, which prove, that the measure serves as a design tool to select preferred processes and to specify the control structure.

2. On Relative Gain Array and Condition Number [4](1992), Article

Jie Chen, James S. Freundenberg, Carl N. Nett

This paper brings a view on deviations in open - loop properties in multivariable systems. The authors focus on different properties of measures of diagonally structured uncertainties. Different worst case estimates are given in terms of Condition number of scaled plant, Relative gain array and block relative gain. The estimates in terms of condition number are shown to be tight. The other two measures only suggest a high deviation when large.

3. Using the μ- Interaction measure in the design and stability analysis of decentralized control structures [13](1989), Article

Nick Hutson, Tse - Wei Wang

Multivariable controllers are the main topic of many researches and theoretical projects, but there are not many multivariable controllers applied in industry. The complexity of multivariable controllers causes computational intensity and other important requirements. Decomposing multivariable systems into a set of SISO problems is one solution to this problem. To do so, an interaction measure between individual inputs and outputs of the system need to be used. The authors propose a μ interaction measure as it predicts stability of the closed - loop full - block system with a diagonal controller, whose design is based on the diagonal plant dynamics. A benefit of this interaction measure is that it measures the lost performance caused by the diagonalization of the plant matrix at the same time. The computation complexity for μ- interaction measure is said to be high, but the authors give three examples for 2x2 systems, where the computation power required is low. The examples are given from chemical process industry and show, that the μ interaction measure analysis approach renders a simple and direct test for the maximum allowable gains of the controller. In contrast to the fact that μ interaction measure guarantees stability of the closed - loop system, it does not mean, that the performance of the delivered combination of inputs and outputs will not bring poor performance.

4. Singular Perturbation for the Dynamic Interaction Measure [12] (1985), Article

Shimizu Kazuyuki, Masakazu Matsubara

Relative Gain Array is appreciated as a good measure of system channels interaction, but its lack of ability to work with dynamic systems is pointed out. Therefore, the authors propose a dynamic extension of RGA for the class of singularity perturbed systems, based on a state space description. The authors give an example from the chemical process industry, where there are many systems with dynamics of different orders. The obtained result is useful for studies of the system dynamics, which show faster responses. These dynamics are often neglected in system design, therefore, the proposed method is well appreciated.

5.Gramian based interaction measure [10](2000), Article

Arthur Conley

A gramian based interaction measure for MIMO systems is introduced by the author of this article. The relative Gain Array (RGA), introduced earlier in [5] is unable to cope with non - minimum phase structures, is insensitive to delays and is evaluated for a particular frequency.

The introduced gramian based methods take advantage of Gramian matrices, which describe controllability and observability properties of stable systems. Using the **P** and **Q** matrices, derived from Lyapunov equations

$$\mathbf{AP} + \mathbf{PA}^T + \mathbf{BB}^T = 0 \tag{4.1}$$

$$\mathbf{A}^T\mathbf{Q} + \mathbf{QA} + \mathbf{C}^T\mathbf{C} = 0 \tag{4.2}$$

are used for calculation of the Hankel singular values, which are singular values of the product of P and Q. The measure can be organized into a scaled participation matrix, described as

$$\phi_{ij} = \frac{\mathrm{trace}[\mathbf{P}_i\mathbf{Q}_i]}{\mathrm{trace}[\mathbf{PQ}]} \leq 1 \ .$$

The main aim of the presented method is to obtain a value of sum of ϕ_{ij} close to one with minimum controller complexity. Analyzing the participation matrix, it is possible to decide on input-output pairings and to assess the benefits of different controller structures.

6. Comparison of some Gramian based interaction measures [11](2008), Article

Björn Halvarsson

The author proposes the squared H_2 norm as a useful interaction measure. This measure can be given various useful energy interpretations, furthermore it can be seen as a measure of the output controllability of the plant. Fundamental properties of the H_2 norm based interaction measure are derived in this article. An important property of this interaction measure is the fact, that it is not affected by time delays whereas the Hankel Interaction Index Array (HIIA) and the Participation Matrix (PM) are.

Different interaction measures are compared in this article with a conclusion, that it could be beneficial to consider different interaction measures and compare the results when solving the pairing problem.

7. The Stability of Multivariable Systems [8] (1972), Article

Howard H. Rosenbrock

The author presents an alternative method to classical existing methods of system stability evaluation. If possible, the stability problem should be decomposed to as many problems, as the number of system inputs is. Closed loop stability can be investigated by closing loop in succession and keeping count of encirclement in each successive loop by the usual single-loop Nyquist criterion.

8. Robust Control of Ill-Conditioned Plants: High- Purity Distillation [3](1988), Article

Sigurd Skogestad, Manfred Morari, John C. Doyle

Properties of multivariable plants are explained with emphasis on system output and input directions. Structured singular value and Relative gain array measures of system conditioning are explained and compared. A chemical system of High Purity Distillation is taken as an example for application of these techniques. System uncertainty modelling, with different uncertainty representation is presented and outlined for the example of High - Purity Distillation.

9. Multivariable Feedback Design: Concepts for Classical/Modern Synthesis [7] (1981), Article

John Doyle, Günter Stein

The article is mainly focused on dealing with uncertainties in multivariable feedback design. The difference between SISO and MIMO properties from the feedback design point of view is described. The stability and performance conditions for MIMO systems are derived, and the fact that there are only small differences comparing to SISO design are pointed out. Methodologies as inverse Nyquist array and characteristic loci are mentioned as examples of decomposing multivariable systems into a sequence of scalar problems. The authors are interested in bringing the theoretical innovations to practice, as they mention, that most of the modern control theory is often not used in practice. An example of LQG regulator design for longitudal control for a CH - 47 tandem rotor helicopter is given for illustration of use of the presented methods.

10. Interaction Analysis and Control Structure Selection in a Wastewater Treatment Plant Model [9](2005), Article

Par Samuelson, Björn Halvarsson, Bengt Carlsson

A comparison between Relative Gain Array (RGA) and Hankel interaction index array (HIIA) is the main topic of this article. Both methods for interaction analysis are explained with their advantages and disadvantages mentioned. RGA does not give a good indication how to choose a control system structure it suggests best possible input-output pairing. The HIIA is scaling dependent, but it considers the controllability and observability of every subsystem in the plant separately, so it is a very good measure of cross - couplings of the system at an operating point. Another advantage is that HIIA takes into account the whole frequency range, but RGA has to be calculated for every single frequency. An example of advantages and disadvantages of both interaction measures is given on analysis of a wastewater treatment plant model.

11. A new input/output constrained Model Predictive Control with frequency domain tuning technique and its application to an ethylene plant [14](1993), Article

Y. Iino, K. Tomida, H. Fujiwara, Y Takagi, T. Shigemasa A. Yamamoto, A.

Model Predictive Control is presented as a process control technique, that attracts attention as a practical approach. It is useful in a higher - level control layer of hierarchical control process. The article proposes a new MPC method, derived from modification of General Predictive Control (GPC). In order to improve robustness of the predictor against noises, a Kalmann filter based predictor is introduced. A new weighting factor is added to the quadratic cost function. This weighting factor is time-dependent and its purpose is to improve transient response characteristics of the predictive controller. Another new factor is added to the cost function in order to reduce the reference tracking error of the manipulation variables. This factor handles redundancy of the manipulation variables, in case there is some.

The main part of the article brings a new parameter tuning method which adjusts the weighting factors in the cost function, considering robust stability of the control system. The authors propose that this method is important, as the plant model includes some uncertainty in general, so the MPC system should be sufficiently robust to be stable. The weighting factors in the cost function strongly influence the stability of the system. The proposed iterative tuning method is based on frequency response analysis.

At the end of the article, the presented Model Predictive method is applied to a model of ethylene plant. The control system with Model predictive controller proves to be better than conventional control used for such plant. The improvements of control performance were in decoupling of strongly coupled process variables, compensation of the delay dynamics and in disturbance rejection.

12. Multi-variable nonlinear MPC of an ill-conditioned distillation column [15] (2004), Article

Jonas B. Waller, Jari M. Böling

The article is mainly focused on quasi-ARMAX modelling and control of multi- variable ill-conditioned and nonlinear processes. MPC controller is said to be useful in the petrochemical industry. As the processes in this field include rapid large frequent disturbances and the operating point changes frequently, it is not possible to ignore nonlinearities, as it is often done in MPC control of other types of processes. A nonlinear MPC controller is used to control such type of systems. A distillation column is used as a practical example of use of NMPC, as it is a nonlinear, ill - conditioned system. The conditioning of the system is observed using minimized condition numbers and relative gain array (RGA). A description of ill - conditioning is given on an example of changing input signal directions while watching the change in the output direction. When the input direction is changed so that it creates a circle, the output signal directions create an ellipse. The more ill - conditioned system, the more oval ellipse. A nonlinearity in system bents the ellipse. The complex system of distillation column is restricted from 38 states to 2 inputs, 2 outputs system, and a NMPC controller is designed using quasi - ARMAX model. The NMPC controller designed by this technique brings a good control of the distillation column at different operating regions.

13. Robust disturbance modelling for model predictive control with application to multivariable ill-conditioned processes [16] (2002), Article

Gabriele Pannocchia

This paper is concerned with the fact, that a disturbance model is used to achieve offset free performance of MPC controller. Most industrial MPC implementations use a constant step disturbance added to the measured process variables. This method is acceptable for stable plants, but it can not be used if the plant is unstable, as the observer contains the open - loop unstable poles. There are different disturbance models discussed in the article. When considering ill - conditioned systems, the inverse controller becomes sensitive to input uncertainties and plant mismatch. MPC algorithms suffer from sensitivity to uncertainties when the process is ill - conditioned, so it is common to increase the control move suppression factor. The authors point out, that a significant improvement in robustness can be accomplished by choosing a suitable disturbance model. An input disturbance model is a more robust choice than the conventional output disturbance model. In the input disturbance model the disturbance is assumed to be a constant step added to the process input. This disturbance model is used for an ill - conditioned system of distillation column with conditioning number 142. This model requires off - line solution of min - max optimization problem, and it shows to be a good choice for the presented distillation column, as there is a significant improvement in robustness comparing to the conventional output disturbance model.

14. Robustness of MPC and Disturbance Models for Multivariable Ill-conditioned Processes [21] (2001), Article

Gabriele Pannocchia, James B. Rawlings

The importance of good disturbance modelling for ill - conditioned systems is emphasized in this article. According to the authors, the widely - used output disturbance model is not the best disturbance representation for ill-conditioned plants, as it is not robust to modelling errors when the process model is ill-conditioned. The input disturbance model is proposed as a better disturbance representation, as it is robust to uncertainties. A method based on min-max optimization is proposed to be used for disturbance modelling in ill-conditioned systems. The advantage of using an input disturbance model is the ability of overcoming a robustness problem of ill - conditioned plant when using output disturbance model and the problem with rejection of slow-dynamics disturbance with output disturbance model. The gained disturbance model is close to the convenient input disturbance model. Practical application of the disturbance modelling is shown on the well - known model of ill - conditioned distillation column.

15. Robustness of Model Predictive Control for Ill-Conditioned Distillation Process [17], (2005), Article

Vu Trieu Minh, Nitin Afzulpurkar

The authors of this article present a method of MPC control of an ill - conditioned plant. A previously used method which deletes some of the system outputs that are controlled with difficulties is presented first. A different approach is proposed by the authors. They propose to change a control goal from tracking a reference signal into keeping the system output in certain regions. This approach uses the property of some systems that it is not always necessary to exactly track a reference. The control goal is looser than the conventional reference tracking, therefore, it is easier to achieve. The MPC cost function is rewritten in terms of regions. The equations used in this approach are presented at the beginning of chapter 3 of this thesis. A distillation column is used for the system behaviour simulation.

16. MIMO interaction measure and controller structure selection [22], (2004), Article

Mario E. Salgado, Arthur Conley

Gramian based interaction measures are studied in details by the authors of this article. The authors describe input / output pairing selection for a system where the inputs and outputs are already given. System Gramian definition for both continuous and discrete systems is given, followed by proposal of a Gramian based interaction measure. This measure is called a participation matrix (PM). PM is a matrix of numbers, built upon a dynamic plant model and it has no limitations regarding the number of plant inputs and outputs in a decentralized control architecture. Some information on PM is given in section 2.2.4 of this thesis.

Appendix B

Contents of the attached CD

The attached CD contains an electronic version of this thesis. The Matlab source codes are treated as confidental and are property of the Honeywell company.

Structure of the atached CD:

- Directory Text: Thesis (.pdf file format)